第八讲. 扩张图与去随机

Randomness and Hardness

Many derandomization results are based on the assumption that certain random/hard objects exist.

Some unconditional derandomization can be achieved using explicit constructions of pseduorandom objects.

Synopsis

1. Basic Linear Algebra
2. Random Walk
3. Expander Graph
4. Explicit Construction of Expander Graph
5. Reingold's Theorem

Basic Linear Algebra

Three Views

All boldface lower case letters denote column vectors.

Matrix = Linear transformation : $\mathbf{Q}^n \to \mathbf{Q}^m$

1. $f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$, $f(c\mathbf{u}) = cf(\mathbf{u})$
2. the matrix M_f corresponding to f has $f(\mathbf{e}_j)$ as the j-th column

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Interpretation of $\mathbf{v} = A\mathbf{u}$

1. Dynamic view: $\mathbf{u}$ is transformed to $\mathbf{v}$, movement in one basis
2. Static view: $\mathbf{u}$ in the column basis is the same as $\mathbf{v}$ in the standard basis, movement of basis

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Equation, Geometry (row picture), Algebra (column picture)

- Linear equation, hyperplane, linear combination

Suppose M is a matrix, $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are column vectors, and $\mathbf{r}_1, \ldots, \mathbf{r}_n$ are row vectors.

$$M(\mathbf{c}_1, \ldots, \mathbf{c}_n) = (M\mathbf{c}_1, \ldots, M\mathbf{c}_n) \tag{1}$$

$$(\mathbf{c}_1, \ldots, \mathbf{c}_n)\begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{pmatrix} = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \ldots + \mathbf{c}_n\mathbf{r}_n \tag{2}$$

Inner Product, Projection, Orthogonality

1. Inner product $\mathbf{u}^\dagger\mathbf{v}$ measures the degree of colinearity of $\mathbf{u}$ and $\mathbf{v}$
   - $\frac{\mathbf{u}}{\|\mathbf{u}\|}$ is the normalization of $\mathbf{u}$
   - $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u}^\dagger\mathbf{v} = 0$
   - $\frac{\mathbf{u}^\dagger\mathbf{v}}{\|\mathbf{u}\|}\frac{\mathbf{u}}{\|\mathbf{u}\|}$ is the projection of $\mathbf{v}$ onto $\mathbf{u}$, where $\|\mathbf{u}\| = \sqrt{\mathbf{u}^\dagger\mathbf{u}}$ is the length of $\mathbf{u}$
   - projection matrix $P = \frac{\mathbf{u}\mathbf{u}^\dagger}{\mathbf{u}^\dagger\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{u}^\dagger}{\|\mathbf{u}\|}$
   - suppose $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent. the projection of $\mathbf{v}$ onto the subspace spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_m$ is $P\mathbf{v}$, where the projection matrix P is $A(A^\dagger A)^{-1}A^\dagger$. if $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are orthonormal, $P = \mathbf{u}_1\mathbf{u}_1^\dagger + \ldots + \mathbf{u}_m\mathbf{u}_m^\dagger = I_m$.
2. Basis, orthonormal basis, orthogonal matrix
3. $Q^{-1} = Q^\dagger$ for every orthogonal matrix Q
   - Gram-Schmidt orthogonalization, $A = QR$

Cauchy-Schwartz Inequality. $\cos\theta = \frac{\mathbf{u}^\dagger\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1.$

Fixpoints for Linear Transformation

We look for fixpoints of a linear transformation $A : \mathbf{R}^n \to \mathbf{R}^n$.

$$A\mathbf{v} = \lambda\mathbf{v}.$$

If there are n linear independent fixpoints $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then every $\mathbf{v} \in \mathbf{R}^n$ is some linear combination $c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$. By linearity,

$$A\mathbf{v} = c_1A\mathbf{v}_1 + \ldots + c_nA\mathbf{v}_n = c_1\lambda_1\mathbf{v}_1 + \ldots + c_n\lambda_n\mathbf{v}_n.$$

If we think of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as a basis, the effect of the transform A is to stretch the coordinates in the directions of the axes.

Eigenvalue, Eigenvector, Eigenmatrix

If $A-\lambda I$ is singular, an eigenvector $\mathbf{x}$ satisfies $\mathbf{x} \neq \mathbf{0}$, $A\mathbf{x}=\lambda\mathbf{x}$; and λ is the eigenvalue.

1. $S = [\mathbf{x_1}, \ldots, \mathbf{x_n}]$ is the eigenmatrix. By definition $AS = S\Lambda$.
2. If $\lambda_1, \ldots, \lambda_n$ are different, $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent.
3. If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent, $A = S\Lambda S^{-1}$.

Suppose $c_1\mathbf{x}_1 + \ldots + c_n\mathbf{x}_n = \mathbf{0}$. Then $c_1\lambda_1\mathbf{x}_1 + \ldots + c_n\lambda_n\mathbf{x}_n = \mathbf{0}$. It follows that $c_1(\lambda_1 - \lambda_n)\mathbf{x}_1 + \ldots + c_{n-1}(\lambda_{n-1} - \lambda_n)\mathbf{x}_{n-1} = \mathbf{0}$. By induction we eventually get $c_1(\lambda_1 - \lambda_2)\ldots(\lambda_1 - \lambda_n)\mathbf{x}_1 = \mathbf{0}$. Thus $c_1 = 0$. Similarly $c_2 = \ldots = c_n = 0$.

- We shall write the spectrum $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$.
- $\rho(A) = |\lambda_1|$ is called spectral radius.

Similarity Transformation

Similarity Transformation = Change of Basis

1. A is similar to B if $A = MBM^{-1}$ for some invertible M.
2. $\mathbf{v}$ is an eigenvector of A iff $M^{-1}\mathbf{v}$ is an eigenvector of B.

A and B describe the same transformation using different bases.

1. The basis of B consists of the column vectors of M.
2. A vector $\mathbf{x}$ in the basis of A is transformed into the vector $M^{-1}\mathbf{x}$ in the basis of B, that is $\mathbf{x} = M(M^{-1}\mathbf{x})$.
3. B then transforms $M^{-1}\mathbf{x}$ into some $\mathbf{y}$ in the basis of B.
4. In the basis of A the vector $A\mathbf{x}$ is $M\mathbf{y}$.

Fact. Similar matrices have the same eigenvalues.

Triangularization

Diagonalization transformation is a special case of similarity transformation. In diagonalization Q provides an orthogonal basis.

Question. Is every matrix similar to a diagonal matrix?

Schur's Lemma. For each matrix A there is a unitary matrix U such that $T = U^{-1}AU$ is triangular. The eigenvalues of A appear in the diagonal of T.

Diagonalization

What are the matrices that are similar to diagonal matrices?

A matrix N is normal if $NN^\dagger = N^\dagger N$.

Theorem. A matrix N is normal iff $T = U^{-1}NU$ is diagonal iff N has a complete set of orthonormal eigenvectors.

Proof.

If N is normal, T is normal. It follows from $T^\dagger = T$ that T is diagonal. If T is diagonal, it is the eigenvalue matrix of N, and $NU = UT$ says that the column vectors of U are precisely the eigenvectors. □

Hermitian Matrix and Symmetric Matrix

	real matrix	complex matrix
length	$\|x\| = \sqrt{\sum_{i\in[n]} x_i^2}$	$\|x\| = \sqrt{\sum_{i\in[n]} \|x_i\|^2}$
conjugate transpose	$A^\dagger$	$A^\dagger$
inner product	$\mathbf{x}^\dagger\mathbf{y} = \sum_{i\in[n]} x_i y_i$	$\mathbf{x}^\dagger\mathbf{y} = \sum_{i\in[n]} \overline{x}_i y_i$
orthogonality	$\mathbf{x}^\dagger\mathbf{y} = 0$	$\mathbf{x}^\dagger\mathbf{y} = 0$
symmetric/Hermitian	$A^\dagger = A$	$A^\dagger = A$
diagonalization	$A = Q\Lambda Q^\dagger$	$A = U\Lambda U^\dagger$
orthogonal/unitary	$Q^\dagger Q = I$	$U^\dagger U = I$

Fact. If $A^\dagger = A$, then $\mathbf{x}^\dagger A\mathbf{x} = (\mathbf{x}^\dagger A\mathbf{x})^\dagger$ is real for all complex $\mathbf{x}$.

Fact. If $A^\dagger = A$, the eigenvalues are real since $\mathbf{v}^\dagger A\mathbf{v} = \lambda\mathbf{v}^\dagger\mathbf{v} = \lambda\|\mathbf{v}\|^2$.

Fact. If $A^\dagger = A$, the eigenvectors of different eigenvalues are orthogonal.

Fact. $\|U\mathbf{x}\|^2 = \|\mathbf{x}\|^2$ and $\|Q\mathbf{x}\|^2 = \|\mathbf{x}\|^2$.

Spectral Theorem

Theorem. Every Hermitian matrix A can be diagonalized by a unitary matrix U. Every symmetric matrix A can be diagonalized by an orthogonal matrix Q.

$$
\begin{aligned}
U^\dagger A U &= \Lambda, \\
Q^\dagger A Q &= \Lambda.
\end{aligned}
$$

The eigenvalues are in Λ; the orthonormal eigenvectors are in Q respectively U.

Corollary. Every Hermitian matrix A has a spectral decomposition.

$$A = U \Lambda U^\dagger \stackrel{(1)(2)}{=} \sum_{i \in [n]} \lambda_i \mathbf{u}_i \mathbf{u}_i^\dagger .$$

Notice that $I = UU^\dagger \stackrel{(2)}{=} \sum_{i \in [n]} \mathbf{u}_i \mathbf{u}_i^\dagger$.

Positive Definite Matrix

Symmetric matrixes with positive eigenvalues are at the center of many applications.

A symmetric matrix A is positive definite if $\mathbf{x}^\dagger A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

Theorem. Suppose A is symmetric. The following are equivalent.

1. $\mathbf{x}^\dagger A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.
2. $\lambda_i > 0$ for all the eigenvalues λ_i.
3. $A = R^\dagger R$ for some matrix R with independent columns.

If we replace $>$ by $\geq$, we get positive semidefinite matrices.

Singular Value Decomposition

Consider an $m \times n$ matrix A. Both $AA^\dagger$ and $A^\dagger A$ are symmetric.

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1. $AA^\dagger$ is positive semidefinite since $\mathbf{x}^\dagger AA^\dagger \mathbf{x} = \|A^\dagger \mathbf{x}\|^2 \geq 0$.
2. $AA^\dagger = U\Sigma' U^\dagger$, where U consists of the orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$ and Σ' is the diagonal matrix made up from the eigenvalues $\sigma_1^2 \geq \ldots \geq \sigma_r^2$.
3. $A^\dagger A = V\Sigma'' V^\dagger$.
4. $AA^\dagger \mathbf{u}_i = \sigma_i^2 \mathbf{u}_i$ implies that $(\sigma_i^2, A^\dagger \mathbf{u}_i)$ is an eigenpair for $A^\dagger A$. So $\mathbf{v}_i = \frac{A^\dagger \mathbf{u}_i}{\|A^\dagger \mathbf{u}_i\|}$.
5. $\mathbf{u}_i^\dagger AA^\dagger \mathbf{u}_i = \mathbf{u}_i^\dagger \sigma_i^2 \mathbf{u}_i = \sigma_i^2$. So $\|A^\dagger \mathbf{u}_i\| = \sigma_i$.
6. $A\mathbf{v}_i = A\frac{A^\dagger \mathbf{u}_i}{\|A^\dagger \mathbf{u}_i\|} = \frac{\sigma_i^2 \mathbf{u}_i}{\sigma_i} = \sigma_i \mathbf{u}_i$.

Hence $AV = U\Sigma$, or $A = U\Sigma V^\dagger$. Notice that Σ an $m \times n$ matrix.

Singular Value Decomposition

We call

1. $\sigma_1, \ldots, \sigma_r$ the singular values of A, and
2. $U\Sigma V^\dagger$ the singular value decomposition, or SVD, of A.

Lemma. If A is normal, then $\sigma_i = |\lambda_i|$ for all $i \in [n]$.

Proof.
Since A is normal, $A = U\Lambda U^\dagger$ by diagonalization. Now $A^\dagger A = AA^\dagger = U\Lambda^2 U^\dagger$. So the spectrum of $A^\dagger A / AA^\dagger$ is $\lambda_1^2, \ldots, \lambda_n^2$. □

Rayleigh Quotient

Suppose A is an $n \times n$ Hermitian matrix, $(\lambda_1, \mathbf{v}_1)$, ..., $(\lambda_n, \mathbf{v}_n)$ are the eigenpairs.

The Rayleigh quotient of A and nonzero $\mathbf{x}$ is defined as follows:

$$R(A, \mathbf{x}) = \frac{\mathbf{x}^\dagger A \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}} = \frac{\sum_{i \in [n]} \lambda_i \|\mathbf{v}_i^\dagger \mathbf{x}\|^2}{\sum_{i \in [n]} \|\mathbf{v}_i^\dagger \mathbf{x}\|^2}. \tag{3}$$

It is clear from (3) that

- if $\lambda_1 \geq \ldots \geq \lambda_n$, then $\lambda_i = \max_{\mathbf{x} \perp \mathbf{v}_1, \ldots, \mathbf{x} \perp \mathbf{v}_{i-1}} R(A, \mathbf{x})$, and
- if $|\lambda_1| \geq \ldots \geq |\lambda_n|$, then $|\lambda_i| = \max_{\mathbf{x} \perp \mathbf{v}_1, \ldots, \mathbf{x} \perp \mathbf{v}_{i-1}} |R(A, \mathbf{x})|$.

One can use Rayleigh quotient to derive lower bound for λ_i.

Vector Norm

The norm of a vector is a measure of its magnitude/size/length.

A norm on $\mathbf{F}^n$ is a function $\|\_\| : \mathbf{F}^n \to \mathbf{R}^{\geq 0}$ satisfying the following:

1. $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = \mathbf{0}$.
2. $\|a\mathbf{v}\| = |a| \cdot \|\mathbf{v}\|$.
3. $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

A vector space with a norm is called a normed vector space.

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1. L^1-norm. $\|\mathbf{v}\|_1 = |\mathbf{v}_1| + \ldots + |\mathbf{v}_n|$.
2. L^2-norm. $\|\mathbf{v}\|_2 = \sqrt{|\mathbf{v}_1|^2 + \ldots + |\mathbf{v}_n|^2} = \sqrt{\mathbf{v}^\dagger \mathbf{v}}$.
3. L^p-norm. $\|\mathbf{v}\|_p = \sqrt[p]{|\mathbf{v}_1|^p + \ldots + |\mathbf{v}_n|^p}$.
4. L^∞-norm. $\|\mathbf{v}\|_\infty = \max\{|\mathbf{v}_1|, \ldots, |\mathbf{v}_n|\}$.

Matrix Norm

We define matrix norm in compatible with vector norm. Suppose $\mathbf{F}^n$ is a normed vector space over field $\mathbf{F}$.

An induced matrix norm is a function $\|\_\| : \mathbf{F}^{n\times n} \to \mathbf{R}^{\geq 0}$ satisfying the following properties.

1. $\|A\| = 0$ iff $A = \mathbf{0}$.
2. $\|aA\| = |a|\cdot\|A\|$.
3. $\|A + B\| \leq \|A\| + \|B\|$.
4. $\|AB\| \leq \|A\|\cdot\|B\|$.

Matrix Norm

A matrix norm measures the amplifying power of a matrix. Define

$$\|A\| = \max_{\mathbf{v}\neq\mathbf{0}} \frac{\|A\mathbf{v}\|}{\|\mathbf{v}\|}.$$

It satisfies (1-4). Additionally $\|A\mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\|$ for all $\mathbf{x}$.

$$\begin{aligned} \|A\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |A_{i,j}|, \\ \|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |A_{i,j}|. \end{aligned}$$

Lemma. $\rho(A) \leq \|A\|$.

Spectral Norm

$\|A\|_2$ is called the spectral norm of A.

$$\frac{1}{\sqrt{n}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1.$$

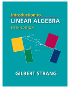
Lemma. $\|A\|_2 = \sigma_1$.

Corollary. If A is a normal matrix, then $\|A\|_2 = |\lambda_1|$.

Let $A^\dagger A = V\Sigma V^\dagger$, let $V = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$, and let $\mathbf{x} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n$. Then

$$\|A\mathbf{x}\|_2^2 = \mathbf{x}^\dagger(A^\dagger A\mathbf{x}) = \mathbf{x}^\dagger(\sum_{i\in[n]} \sigma_i^2 a_i \mathbf{v}_i) \leq \sigma_1^2\|\mathbf{x}\|_2^2.$$

The equality holds when $\mathbf{x} = \mathbf{v}_1$. Therefore $\|A\|_2 = \sigma_1$.



MIT Open Course

https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/

Random Walk

Graphs are the prime objects of study in combinatorics.

The matrix representation of graphs lends itself to an algebraic treatment to these combinatorial objects. It is especially effective in the treatment of regular graph.

Our digraph admit both self-loops and parallel edges. An undirected edge is seen as two directed edges in opposite directions.

In this lecture whenever we say graph, we mean undirected graph.

Random Walk Matrix

The reachability matrix M of a digraph G is defined by $M_{j,i} = 1$ if there is an edge from vertex i to vertex j; $M_{j,i} = 0$ otherwise.

The random walk matrix A of a d-regular digraph G is $\frac{1}{d}M$.

If G is a graph, we will also write G for its random walk matrix!

Stationary Distribution

Let $\mathbf{p}$ be a probability distribution over the vertices of G and A is the random walk matrix of G. Then $A^k\mathbf{p}$ is the distribution after k-step random walk.

$\lim_{k\to\infty} A^k\mathbf{p}$.

Bipartite Graph

Consider the following periodic digraph G.

- The vertices are arranged in n layers.
- Edges are from the i-th layer to the j-th layer, where $j = i + 1 \ mod \ n$.

Does $G^k\mathbf{p}$ converge to a stationary state? What if the edges are undirected?

When $n = 2$, it's the undirected bipartite graph.

Spectral Graph Theory

In spectral graph theory graph properties are characterized by graph spectrums.

Suppose G is a d-regular graph.

1. 1 is an eigenvalue of G and its associated eigenvector $\mathbf{1} = (\frac{1}{n}, \ldots, \frac{1}{n})^\dagger$ is the stationary distribution vector. In other words $G\mathbf{1} = \mathbf{1}$.
2. All eigenvalues have absolute values ≤ 1.
3. G is disconnected if and only if 1 is an eigenvalue of multiplicity at least 2.
4. If G is connected, G is bipartite if and only if -1 is an eigenvalue of G.

In 2 and 3($\Leftarrow$) and 4($\Leftarrow$), consider the entry with the largest absolute value.

Rate of Convergence

For a regular graph G, we define

$$\lambda_G \stackrel{\text{def}}{=} \max_{\mathbf{p}} \frac{\|G\mathbf{p} - \mathbf{1}\|_2}{\|\mathbf{p} - \mathbf{1}\|_2} = \max_{\mathbf{v} \perp \mathbf{1}} \frac{\|G\mathbf{v}\|_2}{\|\mathbf{v}\|_2} = \max_{\mathbf{v} \perp \mathbf{1}, \|\mathbf{v}\|_2 = 1} \|G\mathbf{v}\|_2,$$

where $\mathbf{p}$ is over all probability distribution vectors.

The two definitions are equivalent.

1. $(\mathbf{p} - \mathbf{1}) \perp \mathbf{1}$ and $G\mathbf{p} - \mathbf{1} = G(\mathbf{p} - \mathbf{1})$.
2. For each $\mathbf{v} \perp \mathbf{1}$, $\mathbf{p} = \epsilon\mathbf{v} + \mathbf{1}$ is a probability distribution for a sufficiently small ϵ.

By definition $\|G\mathbf{v}\|_2 \leq \lambda_G \|\mathbf{v}\|_2$ for all $\mathbf{v}$ such that $\mathbf{v} \perp \mathbf{1}$.

Lemma. $\lambda_G = |\lambda_2|$.

Let $\mathbf{v}_2, \ldots, \mathbf{v}_n$ be the eigenvectors corresponding to $\lambda_2, \ldots, \lambda_n$.

Given $\mathbf{x} \perp \mathbf{1}$, let $\mathbf{x} = c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$. Then

$$\begin{aligned} \|G\mathbf{x}\|^2 &= \|\lambda_2 c_2 \mathbf{v}_2 + \ldots + \lambda_n c_n \mathbf{v}_n\|^2 \\ &= \lambda_2^2 c_2^2 \|\mathbf{v}_2\|^2 + \ldots + \lambda_n^2 c_n^2 \|\mathbf{v}_n\|^2 \\ &\leq \lambda_2^2 (c_2^2 \|\mathbf{v}_2\|^2 + \ldots + c_n^2 \|\mathbf{v}_n\|^2) \\ &= \lambda_2^2 \|\mathbf{x}\|^2. \end{aligned}$$

So $\lambda_G^2 \leq \lambda_2^2$. The equality holds since $\|G\mathbf{v}_2\|^2 = \lambda_2^2\|\mathbf{v}_2\|^2$.

Claim. If C is symmetric and $\|C\|_2 \leq 1$ then $\lambda_C \leq 1$.

Proof.

$$\lambda_C = \max_{\mathbf{v}\perp\mathbf{1}} \frac{\|C\mathbf{v}\|_2}{\|\mathbf{v}\|_2} \leq \max_{\mathbf{v}\perp\mathbf{1}} \frac{\|C\|_2\|\mathbf{v}\|_2}{\|\mathbf{v}\|_2} \leq \|C\|_2 \leq 1.$$

□

Claim. $\lambda_{A+B} \leq \lambda_A + \lambda_B$ for symmetric matrices A, B.

Proof.

$$\lambda_{A+B} = \max_{\mathbf{v}\perp\mathbf{1}} \frac{\|(A+B)\mathbf{v}\|_2}{\|\mathbf{v}\|_2} \leq \max_{\mathbf{v}\perp\mathbf{1}} \frac{\|A\mathbf{v}\|_2+\|B\mathbf{v}\|_2}{\|\mathbf{v}\|_2} \leq \lambda_A + \lambda_B.$$

□

The spectral gap γ_G of a graph G is defined by

$$\gamma_G = 1 - \lambda_G.$$

A graph G has spectral expansion γ, where $\gamma \in (0, 1)$, if $\gamma_G \geq \gamma$.

In an expander the spectral expansion provides a bound on the expansion ratio.

Lemma. Let G be an n-vertex regular graph and $\mathbf{p}$ a probability distribution over the vertices of G. Then

$$\|G^\ell \mathbf{p} - \mathbf{1}\|_2 \leq \lambda_G^\ell \|\mathbf{p} - \mathbf{1}\|_2 < \lambda_G^\ell.$$

The first inequality holds because

$$\frac{\|G^\ell \mathbf{p} - \mathbf{1}\|_2}{\|\mathbf{p} - \mathbf{1}\|_2} = \frac{\|G^\ell \mathbf{p} - \mathbf{1}\|_2}{\|G^{\ell-1} \mathbf{p} - \mathbf{1}\|_2} \cdot \frac{\|G^{\ell-1} \mathbf{p} - \mathbf{1}\|_2}{\|G^{\ell-2} \mathbf{p} - \mathbf{1}\|_2} \cdot \ldots \cdot \frac{\|G \mathbf{p} - \mathbf{1}\|_2}{\|\mathbf{p} - \mathbf{1}\|_2} \leq \lambda_G^\ell.$$

The second inequality holds because

$$\|\mathbf{p} - \mathbf{1}\|_2^2 = \|\mathbf{p}\|_2^2 + \|\mathbf{1}\|_2^2 - 2\langle \mathbf{p}, \mathbf{1} \rangle \leq 1 + \frac{1}{n} - 2\frac{1}{n} < 1.$$

In terms of random walk, λ_G bounds the speed of mixing time. [if G is bipartite, $\lambda_G = 1$.]

Lemma. If G is an n-vertex d-regular graph with self-loop at each vertex, $\gamma_G \geq \frac{1}{6dn^2}$.

Let $\mathbf{u}$ be the unit vector such that $\mathbf{u}\perp\mathbf{1}$ and $\lambda_G = \|G\mathbf{u}\|_2$, and let $\mathbf{v} = G\mathbf{u}$.

- If we can prove $1-\|\mathbf{v}\|_2^2 \geq \frac{1}{3dn^2}$, we will get $\lambda_G = \|\mathbf{v}\|_2 \leq 1-\frac{1}{6dn^2}$, hence the lemma.
- It's easy to show $1-\|\mathbf{v}\|_2^2 = \|\mathbf{u}\|_2^2 - \|\mathbf{v}\|_2^2 = \|\mathbf{u}\|_2^2 - 2\langle G\mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|_2^2 = \sum_{i,j} G_{i,j}(\mathbf{u}_i-\mathbf{v}_j)^2$.

Now $\mathbf{u}_i - \mathbf{u}_j \geq \frac{1}{\sqrt{n}}$ for some $i,j\in[n]$. Let $i \to i_1 \to \ldots \to i_k \to j$ be minimal from i to j. Then

$$1/\sqrt{n} \leq \mathbf{u}_i - \mathbf{u}_j \leq |\mathbf{u}_i - \mathbf{v}_i| + |\mathbf{v}_i - \mathbf{u}_{i_1}| + |\mathbf{u}_{i_1} - \mathbf{v}_{i_1}| + \ldots + |\mathbf{v}_{i_k} - \mathbf{u}_j| \tag{4}$$

$$\leq \sqrt{(\mathbf{u}_i-\mathbf{v}_i)^2 + (\mathbf{v}_i-\mathbf{u}_{i_1})^2 + \ldots + (\mathbf{v}_{i_k}-\mathbf{u}_j)^2}\cdot\sqrt{2D}, \tag{5}$$

where D is the diameter of G. Notice that there are $k+1$ edges and k self-loops in (4). Thus

$$1-\|\mathbf{v}\|_2^2 = \sum_{i,j} G_{i,j}(\mathbf{u}_i-\mathbf{v}_j)^2 \geq \frac{1}{d}\cdot\sum_{i,j}(\mathbf{u}_i-\mathbf{v}_j)^2 \geq \frac{1}{d}\cdot\text{red} \geq \frac{1}{d}\cdot\frac{1}{n\cdot(2D)} \geq \frac{1}{3dn^2}$$

using the inequality $2D \leq 3n$.

Randomized Algorithm for Undirected Connectivity

Corollary. Let G be an n-vertex graph with self-loop on every vertex. Let s, t be connected. Let $\ell > 12dn^2\log(n)$ and let X_ℓ denote the vertex distribution after ℓ step random walk from s. Then $\Pr[X_\ell = t] > \frac{1}{2n}$.

Graphs with self-loops are not bipartite. According to the Lemmas,

$$\|G^\ell \mathbf{e}_s - \mathbf{1}\|_2 < \left(1 - \frac{1}{6dn^2}\right)^{6dn^2\log(n^2)} < \frac{1}{n^2}.$$

It follows that $\left(G^\ell \mathbf{e}_s\right)(i) - \frac{1}{n} > -\frac{1}{n^2}$.

If the walk is repeated for $2n^2$ times, the error probability is reduced to below $\frac{1}{2^n}$.

Randomized Algorithm for Undirected Connectivity

Theorem. UPATH (Undirected Connectivity) is in $\mathbf{RL}$.

Every graph can be turned into a non-bipartite regular graph by introducing self-loops.

Can the random algorithm for UPATH be derandomized? Recall that

$$\mathbf{L} \subseteq \mathbf{RL} \subseteq \mathbf{NL}.$$

Expander Graph

Expander graphs, defined by Pinsker in 1973, are sparse and well connected. They behave approximately like complete graphs.

- Sparsity should be understood in an asymptotic sense.

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1. Fan Chung. Spectral Graph Theory. American Mathematical Society, 1997.
2. Hoory, Linial, and Wigderson. Expander Graphs and their Applications. Bulletin of the AMS, **43**, 439-561, 2006.

Well-connectedness can be characterized in a number of manners.

1. Algebraically, expanders are graphs whose second largest eigenvalue is bounded away from 1 by a constant.
2. Combinatorially, expanders are highly connected. Every set of vertices of an expander has a large boundary geometrically.
3. Probabilistically, expanders are graphs in which a random walk converges to the stationary distribution quickly.

Algebraic Property

Intuitively the faster random walk converges, the better the graph is connected.

According to Lemma, the smaller λ_G is, the faster random walk converges to $\mathbf{1}$.

Suppose $d \in \mathbf{N}$ and $\lambda \in (0,1)$ are constants.

A d-regular graph G with n vertices is an (n,d,λ)-graph if $\lambda_G \leq \lambda$.

It follows from a result on page 29 that an (n,d,λ)-graph is connected.

$\{G_n\}_{n\in\mathbf{N}}$ is a (d,λ)-expander graph family if G_n is an (n,d,λ)-graph for all $n \in \mathbf{N}$.

Probabilistic Property

In an expander random walk converges to the uniform distribution in logarithmic steps.

$$\|G^{\log_{\frac{1}{\lambda}}(n)}\mathbf{p} - \mathbf{1}\|_2 < \lambda^{\log_{\frac{1}{\lambda}}(n)} = \frac{1}{n}. \tag{6}$$

In other words, the mixing time of an expander is logarithmic.

It follows from the inequality in (6) that for every $i \in [n]$,

$$\left(G^{\log_{\frac{1}{\lambda}}(n)}\mathbf{p}\right)(i) > 0.$$

Fact. The diameter of an n-vertex expander graph is $\Theta(\log n)$.

Combinatorial Property

Suppose $G = (V, E)$ is an n-vertex d-regular graph.

- Let $\overline{S}$ stand for $V \setminus S$ for $S \subseteq V$.
- Let $E(S, T)$ be the set of edges $i \to j$ with $i \in S$ and $j \in T$.
- Let $\partial S = E(S, \overline{S})$ for $|S| \leq \frac{n}{2}$.

The expansion constant h_G of G is defined as follows:

$$h_G = \min_{0<|S|\leq \frac{n}{2}} \frac{|\partial S|}{|S|}.$$

Suppose ρ is a constant in $(0,1)$. There are $d|S|$ edges emitting from the nodes of S.

An n-vertex d-regular graph G is an (n, d, ρ)-edge expander if $\frac{h_G}{d} \geq \rho$.

Existence of Expander

Theorem. Let $\epsilon > 0$. There exists $d = d(\epsilon)$ and $N \in \mathbf{N}$ such that for every $n > N$ there exists an $(n, d, \frac{1}{2} - \epsilon)$ edge expander.

Cheeger Inequality

Theorem. Let $G = (V, E)$ be a finite, connected, d-regular graph. Then

$$\frac{\gamma_G}{2} \leq \frac{h_G}{d} \leq \sqrt{2\gamma_G}.$$

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1. J. Dodziuk. Difference Equations, Isoperimetric Inequality and Transience of Certain Random Walks. Trans. AMS, 1984.
2. N. Alon and V. Milman. λ_1, Isoperimetric Inequalities for Graphs, and Superconcentrators. J. Comb. Theory, 1985.
3. N. Alon. Eigenvalues and Expanders. Combinatorica, 1986.

$\frac{\gamma_G}{2} \leq \frac{h_G}{d}$

Let S be such that $|S| \leq \frac{n}{2}$ and $\frac{|\partial(S)|}{|S|} = h_G$. Define $\mathbf{x} \perp \mathbf{1}$ by $\mathbf{x}_i = \begin{cases} |\overline{S}|, & i \in S, \\ -|S|, & i \in \overline{S}. \end{cases}$

$$\begin{aligned} \|\mathbf{x}\|_2^2 &= n|S||\overline{S}|, \\ \mathbf{x}^\dagger G \mathbf{x} &= (|\overline{S}|\mathbf{1}_S - |S|\mathbf{1}_{\overline{S}})^\dagger G (|\overline{S}|\mathbf{1}_S - |S|\mathbf{1}_{\overline{S}}) \\ &= \frac{1}{d}\left(|\overline{S}|^2|E(S,S)| + |S|^2|E(\overline{S},\overline{S})| - 2|S||\overline{S}||E(S,\overline{S})|\right) \\ &= \frac{1}{d}\left(dn|S||\overline{S}| - n^2|E(S,\overline{S})|\right), \end{aligned}$$

where $=$ is due to $d|S| = |E(S,\overline{S})| + |E(S,S)|$ and $d|\overline{S}| = |E(\overline{S},S)| + |E(\overline{S},\overline{S})|$.

The Rayleigh quotient $R(G,\mathbf{x})$ provides a lower bound for λ_G, notice that $|\overline{S}| \geq \frac{n}{2}$.

$$\lambda_G \geq \frac{\mathbf{x}^\dagger G\mathbf{x}}{\|\mathbf{x}\|_2^2} = \frac{1}{d}\frac{dn|S||\overline{S}| - n^2|E(S,\overline{S})|}{n|S||\overline{S}|} = 1 - \frac{1}{d}\cdot\frac{n}{|\overline{S}|}\cdot\frac{|\partial(S)|}{|S|} \geq 1 - \frac{2h_G}{d}.$$

$\frac{h_G}{d} \le \sqrt{2\gamma_G}$

Let $\mathbf{u}\perp\mathbf{1}$ be such that $G\mathbf{u} = \lambda_2\mathbf{u}$. Write $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ respectively $\mathbf{w}$ is defined from $\mathbf{u}$ by replacing the negative respectively positive components by 0.

Wlog, assume that the number of positive components of $\mathbf{v}$ is $\le \frac{n}{2}$.

Wlog, assume that the coordinates of $\mathbf{v}$ satisfy $\mathbf{v}_1 \ge \mathbf{v}_2 \ge \ldots \ge \mathbf{v}_n$. Then

$$\sum_{i,j} G_{i,j}|\mathbf{v}_i^2 - \mathbf{v}_j^2| = 2\sum_{i<j} G_{i,j}\sum_{k=i}^{j-1}(\mathbf{v}_k^2 - \mathbf{v}_{k+1}^2) = 2\sum_{i=1}^{n/2}\sum_{j=i+1}^{n/2} G_{i,j}\sum_{k=i}^{j-1}(\mathbf{v}_k^2 - \mathbf{v}_{k+1}^2) \tag{7}$$

$$= \frac{2}{d}\sum_{k=1}^{n/2}|\partial[k]|(\mathbf{v}_k^2 - \mathbf{v}_{k+1}^2) \ge \frac{2}{d}\sum_{k=1}^{n/2} h_G k(\mathbf{v}_k^2 - \mathbf{v}_{k+1}^2) = \frac{2h_G}{d}\|\mathbf{v}\|_2^2.$$

The equality $=$ is valid because $\mathbf{v}_k = 0$ for all $k > n/2$.

$\frac{h_G}{d} \leq \sqrt{2\gamma_G}$

$\langle G\mathbf{v}, \mathbf{v}\rangle \geq \langle G\mathbf{v}, \mathbf{v}\rangle + \langle G\mathbf{w}, \mathbf{v}\rangle = \lambda_2 \|\mathbf{v}\|_2^2$ because $G\mathbf{u} = \lambda_2 \mathbf{u}$, $\langle \mathbf{w}, \mathbf{v}\rangle = 0$ and $\langle G\mathbf{w}, \mathbf{v}\rangle \leq 0$.

$$1 - \lambda_G \geq 1 - \frac{\langle G\mathbf{v}, \mathbf{v}\rangle}{\|\mathbf{v}\|_2^2} = \frac{\|\mathbf{v}\|_2^2 - \langle G\mathbf{v}, \mathbf{v}\rangle}{\|\mathbf{v}\|_2^2} = \frac{\sum_{i,j} G_{i,j}(\mathbf{v}_i - \mathbf{v}_j)^2}{2\|\mathbf{v}\|_2^2}. \tag{8}$$

See page 34 for the second equality in (8). Using Cauchy-Schwartz Inequality,

$$\sum_{i,j} G_{i,j}(\mathbf{v}_i - \mathbf{v}_j)^2 \cdot \sum_{i,j} G_{i,j}(\mathbf{v}_i + \mathbf{v}_j)^2 \geq \left(\sum_{i,j} G_{i,j} |\mathbf{v}_i^2 - \mathbf{v}_j^2| \right)^2. \tag{9}$$

Now $\langle G\mathbf{v}, \mathbf{v}\rangle \leq \lambda_1 \|\mathbf{v}\|_2^2 = \|\mathbf{v}\|_2^2$. Therefore

$$2\|\mathbf{v}\|_2^2 \cdot \sum_{i,j} G_{i,j}(\mathbf{v}_i + \mathbf{v}_j)^2 \leq 2\|\mathbf{v}\|_2^2 \cdot (2\|\mathbf{v}\|_2^2 + 2\langle G\mathbf{v}, \mathbf{v}\rangle) \leq 8\|\mathbf{v}\|_2^4. \tag{10}$$

(7)+(8)+(9)+(10) implies $\sqrt{2(1-\lambda_G)} \geq \frac{h_G}{d}$.

Combinatorial definition and algebraic definition are equivalent.

---

1. The inequality $\frac{1-\lambda_G}{2} \leq \frac{h_G}{d}$ implies that if G is an (n, d, λ)-expander graph, then it is an $(n, d, \frac{1-\lambda}{2})$ edge expander.
2. The inequality $\frac{h_G}{d} \leq \sqrt{2(1-\lambda_G)}$ implies that if G is an (n, d, ρ) edge expander, then it is an $(n, d, 1-\frac{\rho^2}{2})$-expander graph.

Vertex Expander

Let $N(S)$ be the set of neighbors of the vertex set S. Let $\alpha \in (0, 1)$.

An n-vertex d-degree regular graph G is an (n, α, d, A)-vertex expander iff $N(S) \geq A \cdot |S|$ for all S satisfying $|S| \leq \alpha n$, where $0 < \alpha < 1$.

If the inequality is "$N(S) \geq A \cdot |N(S) \setminus S|$", one gets (n, α, d, A)-boundary expander.

Theorem. If G is a (n, d, λ)-expander, G is a $(n, \alpha, d, \frac{1}{(1-\alpha)\lambda^2+\alpha})$-vertex expander for all $\alpha < 1$.

For $|S| \leq \alpha n$ let π_S be the uniform distribution on S. By definition $\|\pi_S\| = \frac{1}{\sqrt{|S|}}$.

By Cauchy-Schwartz inequality,

$$1 = \sum_{i \in [n]} (G\pi_S)(i) \leq \sqrt{|N(S)|} \cdot \sqrt{\sum_{i \in [n]} ((G\pi_S)(i))^2} = \sqrt{|N(S)|} \cdot \|G\pi_S\|.$$

Observe that $(\pi_S - \mathbf{1}) \perp \mathbf{1}$, and consequently $(G\pi_S - \mathbf{1}) \perp \mathbf{1}$. It follows from 勾股定理 that

$$\frac{1}{|N(S)|} - \frac{1}{n} \leq \|G\pi_S\|^2 - \|\mathbf{1}\|^2 = \|G(\pi_S - \mathbf{1})\|^2 \leq \lambda^2 \cdot \left(\|\pi_S\|^2 - \|\mathbf{1}\|^2\right) = \lambda^2 \cdot \left(\frac{1}{|S|} - \frac{1}{n}\right).$$

Therefore $\frac{|S|}{|N(S)|} \leq \lambda^2 + (1 - \lambda^2)\frac{|S|}{n} \leq \lambda^2 + (1 - \lambda^2)\alpha = (1 - \alpha)\lambda^2 + \alpha$.

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1. M. Tanner. Explicit concentrators from generalized N-gons. SIAM Journal on Algebraic Discrete Methods, 5(3):287–293, 1984.

Convergence in Entropy

Rényi 2-Entropy:

$$H_2(\mathbf{p}) = \log\left(\frac{1}{\|\mathbf{p}\|_2^2}\right).$$

Fact. If G is an (n, d, λ)-expander, then $H_2(G\mathbf{p}) \geq H_2(\mathbf{p})$. The equality holds if and only if $\mathbf{p}$ is uniform.

Proof.

Let $\mathbf{p} = \mathbf{1} + \mathbf{w}$. Then $\mathbf{w} \perp \mathbf{1}$ and $\langle G\mathbf{w}, G\mathbf{1}\rangle = \langle G\mathbf{w}, \mathbf{1}\rangle = \mathbf{w}^\dagger G^\dagger \mathbf{1} = \mathbf{w}^\dagger \mathbf{1} = 0$. Therefore

$$\|G\mathbf{p}\|_2^2 = \|\mathbf{1}\|_2^2 + \|G\mathbf{w}\|_2^2 \leq \|\mathbf{p}\|_2^2 - \|\mathbf{w}\|_2^2 + \lambda^2\|\mathbf{w}\|_2^2 = \left(1 - \frac{\|\mathbf{w}\|_2^2}{\|\mathbf{p}\|_2^2} + \lambda^2\frac{\|\mathbf{w}\|_2^2}{\|\mathbf{p}\|_2^2}\right) \cdot \|\mathbf{p}\|_2^2.$$

The inequality $H_2(G\mathbf{p}) \geq H_2(\mathbf{p})$ then follows. The equality holds when $\mathbf{p} = \mathbf{1}$. □

Random walks increase randomness.

The smaller the spectral gap, or the larger the spectral expansion, the more expander graphs behave like random graphs. This is what the next lemma says.

Expander Mixing Lemma

Lemma. Let $G = (V, E)$ be an (n, d, λ)-expander graph. Let $S, T \subseteq V$. Then

$$\left| |E(S, T)| - \frac{d}{n}|S||T| \right| \le \lambda d \sqrt{|S||T|}. \tag{11}$$

Notice that (11) implies

$$\left| \frac{|E(S, T)|}{dn} - \frac{|S|}{n} \cdot \frac{|T|}{n} \right| \le \lambda. \tag{12}$$

The edge density ≈ the product of the vertex densities. This property is called mixing.

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1. N. Alon and F. Chung. Explicit Construction of Linear Sized Tolerant Networks. Discrete Mathematics, 1988.

Proof of Expander Mixing Lemma

Let $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be the (orthonormal) eigenmatrix of G. So $\mathbf{v}_1 = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})^\dagger$.
Let $\mathbf{1}_S = \sum_i \alpha_i \mathbf{v}_i$ and $\mathbf{1}_T = \sum_j \beta_j \mathbf{v}_j$ be the characteristic vectors of S, T respectively.

$$|E(S,T)| = (\mathbf{1}_S)^\dagger (dG) \mathbf{1}_T = \left(\sum_i \alpha_i \mathbf{v}_i\right)^\dagger (dG) \left(\sum_j \beta_j \mathbf{v}_j\right) = \sum_i d\lambda_i \alpha_i \beta_i.$$

Since $\alpha_1 = (\mathbf{1}_S)^\dagger \mathbf{v}_1 = \frac{|S|}{\sqrt{n}}$ and $\beta_1 = (\mathbf{1}_T)^\dagger \mathbf{v}_1 = \frac{|T|}{\sqrt{n}}$, by Cauchy-Schwartz Inequality,

$$\left| |E(S,T)| - \frac{d}{n}|S||T| \right| = \left| \sum_{i=2}^n d\lambda_i \alpha_i \beta_i \right| \leq d\lambda \sum_{i=2}^n |\alpha_i \beta_i| \leq d\lambda \|\alpha\|_2 \|\beta\|_2.$$

Finally observe that $\|\alpha\|_2 = \|\mathbf{1}_S\|_2 = \sqrt{|S|}$ and $\|\beta\|_2 = \|\mathbf{1}_T\|_2 = \sqrt{|T|}$.

Error Reduction for Random Algorithm

Suppose $A(x, r)$ is a random algorithm with error probability $1/3$. The algorithm uses $r(n)$ random bits on input x with $|x| = n$.

1. Reduce the error probability exponentially by repeating the algorithm $t(n)$ times.
2. Altogether $r(n)t(n)$ random bits are used.

The goal is to achieve the same error reduction rate using far fewer random bits, in fact $r(n) + O(t(n))$ random bits.

The key observation is that a t-step random walk in an expander graph looks like t vertices sampled uniformly and independently.

- Confer the inequality (12).

K_n is perfect from the viewpoint of random walk.

- No matter what distribution it starts with, random walk reaches the uniform distribution in one step.

Let $J_n = [\mathbf{1}, \ldots, \mathbf{1}]$ be the random walk matrix of K_n with self-loop.

Decomposition for Random Walk on Expander

Lemma. Suppose G is an (n, d, λ)-expander. Then $G = (1-\lambda)J_n + \lambda E$ for some E such that $\|E\| \leq 1$.

We may think of a random walk on an expander as a convex combination of two random walks of different type:

- with probability $1-\lambda$ it walks randomly on a complete graph, and
- with probability λ it walks randomly according to an error matrix that does not amplify the distance to the uniform distribution.

Decomposition for Random Walk on Expander

We need to prove that $\|E\mathbf{v}\|_2 \leq \|\mathbf{v}\|_2$ for all $\mathbf{v}$, where E is defined by

$$E = \frac{1}{\lambda}(G - (1-\lambda)J_n).$$

The following proof methodology should now be familiar.

- Let $\alpha = \sum_{i\in[n]} v_i$. Then $\mathbf{v} = \alpha\mathbf{1} + \mathbf{w}$ with $\mathbf{w} \perp \mathbf{1}$.
- $G\mathbf{1} = \mathbf{1}$ and $J_n\mathbf{1} = \mathbf{1}$. Consequently $E(\alpha\mathbf{1}) = \alpha\mathbf{1}$.
- $J_n\mathbf{w} = \mathbf{0}$, hence $E\mathbf{w} = \frac{1}{\lambda}G\mathbf{w}$. Also $G\mathbf{w} \perp \alpha\mathbf{1}$.
- $\|G\mathbf{w}\|_2 \leq \lambda\|\mathbf{w}\|_2$.

Thus $\|E\mathbf{v}\|_2^2 = \|\alpha\mathbf{1} + \frac{1}{\lambda}G\mathbf{w}\|_2^2 = \|\alpha\mathbf{1}\|_2^2 + \|\frac{1}{\lambda}G\mathbf{w}\|_2^2 \leq \|\alpha\mathbf{1}\|_2^2 + \|\mathbf{w}\|_2^2 = \|\mathbf{v}\|_2^2$.

Expander Random Walk Theorem

Theorem. Let G be an (n, d, λ) expander graph, and let $B \subseteq [n]$ satisfy $|B| \leq \beta n$ for some $\beta \in (0, 1)$. Let X_1 be a random variable denoting the uniform distribution on $[n]$ and let X_k be a random variable denoting a $k-1$ step random walk from X_1. Then

$$\Pr\left[\bigwedge_{i\in[k]} X_i \in B\right] \leq \left((1-\lambda)\sqrt{\beta}+\lambda\right)^{k-1}.$$

Expander Random Walk Theorem

Let B_i stand for $X_i \in B$. We need to bound

$$\Pr\left[\bigwedge_{i\in[k]} X_i \in B\right] = \Pr[B_1 \ldots B_k] = \Pr[B_1]\cdot\Pr[B_2|B_1]\ldots\Pr[B_k|B_1\ldots B_{k-1}]. \tag{13}$$

By seeing B as a diagonal matrix, we define the distribution vector $\mathbf{p}_i$ by

$$\mathbf{p}_i = \frac{BG}{\Pr[B_i|B_1\ldots B_{i-1}]}\cdot\ldots\cdot\frac{BG}{\Pr[B_2|B_1]}\cdot\frac{B\mathbf{1}}{\Pr[B_1]},$$

where $\frac{BG}{\Pr[B_2|B_1]}\cdot\frac{B\mathbf{1}}{\Pr[B_1]}$ for example is the normalization of $BG\cdot\frac{B\mathbf{1}}{\Pr[B_1]}$. So the probability in (13) is bounded by $\|(BG)^{k-1}B\mathbf{1}\|_1$. We will prove

$$\|(BG)^{k-1}B\mathbf{1}\|_2 \leq \frac{1}{\sqrt{n}}\left((1-\lambda)\sqrt{\beta}+\lambda\right)^{k-1}.$$

Expander Random Walk Theorem

Using Lemma,

$$\begin{aligned}\|BG\| &= \|B((1-\lambda)J_n+\lambda E)\| \le (1-\lambda)\|BJ_n\| + \lambda\|BE\| = (1-\lambda)\sqrt{\beta}+\lambda\|BE\| \\ &\le (1-\lambda)\sqrt{\beta}+\lambda\|B\|\cdot\|E\| \le (1-\lambda)\sqrt{\beta}+\lambda.\end{aligned}$$

Therefore

$$\|(BG)^{k-1}B\mathbf{1}\|_2 \le \|BG\|_2^{k-1}\cdot\|B\mathbf{1}\|_2 \le \frac{\sqrt{\beta}}{\sqrt{n}}\left((1-\lambda)\sqrt{\beta}+\lambda\right)^{k-1} \le \frac{1}{\sqrt{n}}\left((1-\lambda)\sqrt{\beta}+\lambda\right)^{k-1}.$$

Suppose $\|\mathbf{v}\|_2 = 1$ and $\alpha = \sum_{i\in[n]} v_i$. Then $\mathbf{v} = \alpha\mathbf{1}+\mathbf{w}$ and $\mathbf{w}\perp\mathbf{1}$ and $\alpha \le \sqrt{n}$. Now

- $\|BJ_n\mathbf{v}\|_2 = \|BJ_n\alpha\mathbf{1}\|_2 = \alpha\|B\mathbf{1}\|_2 \le \sqrt{n}\|B\mathbf{1}\|_2 = \sqrt{n}\cdot\frac{\sqrt{\beta}}{\sqrt{n}} = \sqrt{\beta}$, and consequently
- $\|BJ_n\| = \max\{\|BJ_n\mathbf{v}\|_2 \mid \|\mathbf{v}\|_2 = 1\} = \sqrt{\beta}$.

The equality holds when $\mathbf{v} = \left(\frac{1}{\sqrt{n}},\ldots,\frac{1}{\sqrt{n}}\right)^{\dagger}$.

Error Reduction for **RP**

Suppose $\mathbb{A}(x, r)$ is a random algorithm with error probability β.

Let B be the set of r's for which $\mathbb{A}$ errs on x.

Choose an explicit $(2^{|r(|x|)|}, d, \lambda)$-graph $G = (V, E)$ with $V = \{0, 1\}^{|r(|x|)|}$.

Algorithm $\mathbb{B}_1$.

1. Pick $v_0 \in_{\mathrm{R}} V$.
2. Generate a random walk $v_0, \ldots, v_t$.
3. Output $\bigvee_{i=0}^{t} \mathbb{A}(x, v_i)$.

By the Theorem, the error probability of $\mathbb{B}_1$ is no more than $\left((1-\lambda)\sqrt{\beta} + \lambda\right)^{t-1}$.

Error Reduction for **BPP**

Algorithm $\mathbb{B}_2$.

1. Pick $v_0 \in_{\mathrm{R}} V$.
2. Generate a random walk $v_0, \ldots, v_t$.
3. Output $Maj\{\mathbb{A}(x, v_i)\}_{i\in[t]}$.

Fix a set of indices $K \subseteq \{0, 1, \ldots, t\}$ such that $|K| \geq \frac{t+1}{2}$.

$$\Pr[\forall i \in K. v_i \in B] \leq \left((1-\lambda)\sqrt{\beta}+\lambda\right)^{|K|-1} \leq \left((1-\lambda)\sqrt{\beta}+\lambda\right)^{\frac{t-1}{2}} \leq \left(\frac{1}{4}\right)^{t-1},$$

assuming $(1-\lambda)\sqrt{\beta}+\lambda \leq 1/16$. Applying union bound on the choices of K,

$$\Pr[\mathbb{B}_2 \text{ fails}] \leq 2^t \left(\frac{1}{4}\right)^{t-1} = O(2^{-t}).$$

Explicit Construction of Expander Graph

Explicit Construction

If random strings are of log size, explicit expander family is good enough.

- An expander family $\{G_n\}_{n\in\mathbf{N}}$ is explicit if there is a P-time algorithm that outputs the random walk matrix of G_n whenever the input is 1^n. [poly(n).]

If random strings are of polynomial size, strongly explicit expander family is necessary.

- An expander family $\{G_n\}_{n\in\mathbf{N}}$ is strongly explicit if there is a P-time algorithm that on input $\langle n, v, i\rangle$ outputs the index of the i-th neighbor of v in G_n. [polylog(n).]

We will look at several graph product operations. We then show how to use these operations to construct explicit expander graphs.

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1. O. Reingold, S. Vadhan, and A. Wigderson. Entropy Waves, the Zig-Zag Graph Product, and New Constant-Degree Expanders and Extractors. FOCS, 2000.

Path Product

Suppose G, G' are n-vertex graphs (sharing the same set of vertexes) with degree d respectively d'. The path product $G'G$ is defined by the random walk matrix $G'G$.

- $G'G$ is n-vertex dd'-degree.

Lemma. $\lambda_{G'G} \leq \lambda_{G'}\lambda_{G}$.

Proof.
$\lambda_{G'G} = \max_{\mathbf{v}\perp\mathbf{1}} \frac{\|G'G\mathbf{v}\|_2}{\|\mathbf{v}\|_2} = \max_{\mathbf{v}\perp\mathbf{1}} \frac{\|G'G\mathbf{v}\|_2}{\|G\mathbf{v}\|_2} \cdot \frac{\|G\mathbf{v}\|_2}{\|\mathbf{v}\|_2} \leq \max_{\mathbf{v}\perp\mathbf{1}} \frac{\|G'G\mathbf{v}\|_2}{\|G\mathbf{v}\|_2} \cdot \max_{\mathbf{v}\perp\mathbf{1}} \frac{\|G\mathbf{v}\|_2}{\|\mathbf{v}\|_2} \leq \lambda_{G'}\lambda_{G}$ using the fact that $G\mathbf{v}\perp\mathbf{1}$ whenever $\mathbf{v}\perp\mathbf{1}$. □

Lemma. $\lambda_{G^k} = (\lambda_G)^k$.

Proof.
$(\lambda_G)^k$ is the second largest eigenvalue of G^k. □

Tensor Product

Suppose G is an n-vertex d-degree graph and G' is an n'-vertex d'-degree graph. The random walk matrix of the tensor product $G \otimes G'$ is nn'-vertex dd'-degree.

$$G \otimes G' = \begin{pmatrix} a_{11}G' & a_{12}G' & \cdots & a_{1n}G' \\ a_{21}G' & a_{22}G' & \cdots & a_{2n}G' \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}G' & a_{n2}G' & \cdots & a_{nn}G' \end{pmatrix}.$$

$(u, u') \to (v, v')$ in $G \otimes G'$ iff $u \to v$ in G and $u' \to v'$ in G'.

Tensor Product

Lemma. $\lambda_{G \otimes G'} = \max\{\lambda_G, \lambda_{G'}\}$.

If $(\lambda, \mathbf{v})$ is an eigenvpair of G and $(\lambda', \mathbf{v}')$ is an eigenpair of G', then $(\lambda\lambda', \mathbf{v} \otimes \mathbf{v}')$ is an eigenpair of $G \otimes G'$.

Rotation Matrix

Let G be the random walk matrix of an n-vertex regular graph G of degree D.

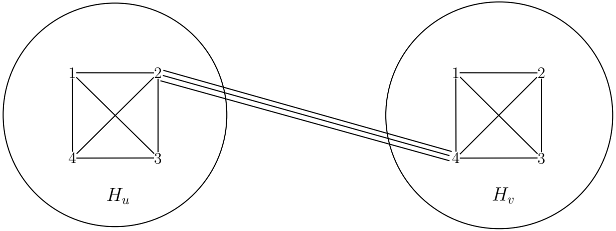
The rotation matrix $\widehat{G}$ is an $(nD)\times(nD)$ adjacent matrix such that $\widehat{G}_{(v,j),(u,i)} = 1$ if

- v is the i-th neighbor of u, and u is the j-th neighbor of v.

Replacement Product

The replacement product $G \circledR H$ is the $2d$-degree graph defined by $\frac{1}{2}\widehat{G} + \frac{1}{2}(I_n \otimes H)$.

- G is an n-vertex regular graph of degree D, and H is a D-vertex regular graph of degree d.



If $\widehat{G}(u, l) = (v, m)$, place d parallel edges from the l-th vertex of H_u to the m-th vertex of H_v.

Lemma. $\lambda_{G \circledR H} \leq 1 - \frac{(1-\lambda_G)(1-\lambda_H)^2}{24}$ and $\gamma_{G \circledR H} \geq \frac{1}{24}\gamma_G \gamma_H^2$.

$$
\begin{aligned}
(G \circledR H)^3 &= \left(\frac{1}{2}\widehat{G} + \frac{1}{2}(I_n \otimes H)\right)^3 = \left(\frac{1}{2}\widehat{G} + \frac{1}{2}\left(I_n \otimes (\lambda_H E + \gamma_H J_D)\right)\right)^3 \\
&= \frac{1}{8}\left(\widehat{G} + \lambda_H(I_n \otimes E) + \gamma_H(I_n \otimes J_D)\right)^3 = \frac{1}{8}\left(\widehat{G}^3 + \ldots + \gamma_H^2(I_n \otimes J_D)\widehat{G}(I_n \otimes J_D)\right) \\
&= \frac{1}{8}\left(\widehat{G}^3 + \ldots + \gamma_H^2(G \otimes J_D)\right),
\end{aligned}
$$

where the last equality is due to Lemma (next slide). Gpplying Lemma and the Claims, we get

$$
(\lambda_{G \circledR H})^3 = \lambda_{(G \circledR H)^3} \leq 1 - \frac{\gamma_H^2}{8} + \frac{\gamma_H^2}{8}\lambda_{G \otimes J_D} \leq 1 - \frac{\gamma_H^2}{8} + \frac{\gamma_H^2}{8}\lambda_G = 1 - \frac{\gamma_H^2}{8}\gamma_G.
$$

We have proved that $(\lambda_{G \circledR H})^3 \leq 1 - \frac{\gamma_G \gamma_H^2}{8} \leq \left(1 - \frac{\gamma_G \gamma_H^2}{24}\right)^3$. Hence $\gamma_{G \circledR H} \geq \frac{1}{24}\gamma_G \gamma_H^2$.

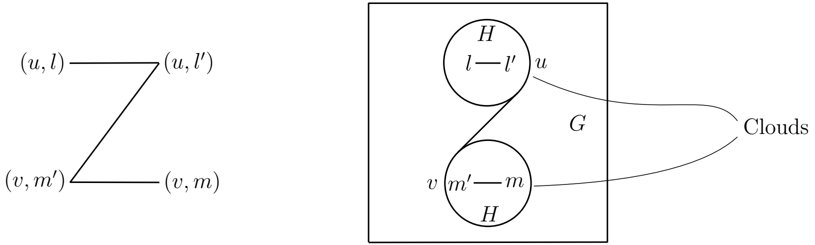
Lemma. $(I_n \otimes J_D)\widehat{G}(I_n \otimes J_D) = G \otimes J_D$.

$$\left((I_n \otimes J_D)\widehat{G}(I_n \otimes J_D)\right)_{(v,m),(u,l)} = \frac{1}{D}\cdot 1 \cdot \frac{1}{D} = \frac{1}{D}\cdot\frac{1}{D} = (G \otimes J_D)_{(v,m),(u,l)}.$$

Zig-Zag Product

The zig-zag product $G \textcircled{z} H$ is the nD-vertex d^2-degree graph $(I_n \otimes H)\widehat{G}(I_n \otimes H)$.

- G is an n-vertex regular graph of degree D, and H is a D-vertex regular graph of degree d.



(v, m) is the (i, j)-th neighbor of (u, l): l' is the i-th neighbor of l in H; v is the l'-th neighbor of u and u is the m'-th neighbor of v; m is the j-th neighbor of m' in H.

Zig-Zag Product

Lemma. $\lambda_{G \textcircled{z} H} \leq \lambda_G + 2\lambda_H$ and $\gamma_{G \textcircled{z} H} \geq \gamma_G \gamma_H^2$.

$\widehat{G}$ is the $(nD)\times(nD)$ rotation matrix of G.

$H = (1-\lambda_H)J_D + \lambda_H E$ for some E with $\|E\|_2 \leq 1$, which is the Lemma. Now

$$\begin{aligned} G \textcircled{z} H &= (I_n \otimes H)\widehat{G}(I_n \otimes H) = ((1-\lambda_H)I_n \otimes J_D + \lambda_H I_n \otimes E)\ \widehat{G}\ ((1-\lambda_H)I_n \otimes J_D + \lambda_H I_n \otimes E) \\ &= (1-\lambda_H)^2 (I_n \otimes J_D)\widehat{G}(I_n \otimes J_D) + \ldots = (1-\lambda_H)^2 (G \otimes J_D) + \ldots, \end{aligned}$$

where $=$ is due to Lemma. Using Lemma and the Claims, one gets

$$\lambda_{G \textcircled{z} H} \leq (1-\lambda_H)^2 \lambda_{G \otimes J_D} + 1 - (1-\lambda_H)^2 \leq \max\{\lambda_G, \lambda_{J_D}\} + 2\lambda_H = \lambda_G + 2\lambda_H.$$

For the inequality $\gamma_{G \textcircled{z} H} \geq \gamma_G \gamma_H^2$, consider $1 - \leq$.

Comment on Zig-Zag Product

1. Typically $d \ll D$.
2. A t-step random walk uses $O(t \log d)$ rather than $O(t \log D)$ random bits.
3. The last lemma is useful when both λ_G and λ_H are small. If not, a different upper bound can be derived. Both upper bounds are discussed in the following paper.

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1. O. Reingold, S. Vadhan, and A. Wigderson. Entropy Waves, the Zig-Zag Graph Product, and New Constant Degree Expanders and Extractors. FOCS, 2000.

We can build up an expander family inductively using the product operations.

	Size	Degree	Expansion
Path Product	$-$	$\uparrow$	$\Uparrow$
Tensor Product	$\uparrow$	$\uparrow$	$\downarrow$
Zigzag Product	$\uparrow$	$\Downarrow$	$\downarrow$

- Use path product and zig-zag product to produce expander family.
- Use constant graph to build constant degree graph family.

To start with we need an expander that can be constructed algorithmically.

Explicit Expander

Suppose $\mathbb{F}$ is a finite field with F elements, where F is a prime power. The F-degree regular graph $G_{\mathbb{F}}$ is defined as follows:

1. The vertex set is $\mathbb{F}\times\mathbb{F}$.
2. There is an edge between (a, b) and (c, d) iff $ac = b + d$.

Consider the path product $G_{\mathbb{F}}^2$. The number of edges between $(a_1, b_1), (a_2, b_2)$ is the number of vertexes shared by the line segments $y = a_1x - b_1$ and 段 $y = a_2x - b_2$.

1. If $a_1 = a_2$ and $b_1 \neq b_2$, there is no shared vertex.
2. If $a_1 = a_2$ and $b_1 = b_2$, there are F shared vertexes.
3. If $a_1 \neq a_2$, there is one shared vertex.

Explicit Expander

Let I_F be the $(F\times F)$-diagonal matrix, and J_F the $(F\times F)$-matrix whose entries are all 1.

$$\text{random walk matrix } A_{\mathbb{F}} \text{ of } G_{\mathbb{F}}^2 = \frac{1}{F^2}\begin{pmatrix} FI_F & J_F & \dots & J_F & J_F \\ J_F & FI_F & \dots & J_F & J_F \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ J_F & J_F & \dots & J_F & FI_F \end{pmatrix} = \frac{I_F \otimes FI_F + (J_F - I_F)\otimes J_F}{F^2}.$$

- The eigenvalue of J_F are F and 0 (with multiplicity $F-1$);
- The eigenvalue of $J_F - I_F$ are $F-1$ and -1 (with multiplicity $F-1$).
- The eigenvalues of $(J_F - I_F)\otimes J_F$ are $(F-1)F$ (multiplicity 1), $-F$, 0.
- The role of $I_F \otimes FI_F$ is to add F to $(F-1)F$, $-F$, 0.
- Conclusion: $G_{\mathbb{F}}^2$ is an $\left(F^2, F^2, \frac{1}{F}\right)$-expander.

Explicit Expander

Theorem. $G_{\mathbb{F}}$ is an $\left(F^2, F, \frac{1}{\sqrt{F}}\right)$-expander.

Proof.

In $G_{\mathbb{F}}$ the i-th neighbor of (a, b) is defined as follows: If $a \neq 0$ and $i \neq 0$, the i-th neighbor is $(i/a, i - b)$; otherwise the i-th neighbor is $(0, -b)$. □

Explicit Expander

1. Construct a $(D^6, D, 1/4)$-expander from $G_{\mathbb{F}}$.
2. Construct a $\left(D^{4k}, D, \frac{1}{2}\right)$-expander family from $G_{\mathbb{F}}$.

Expander Construction I

Let H be a $(D^6, D, 1/4)$-graph constructed from $G_{\mathbb{F}}$. Define

$$\begin{aligned} G_0 &= H^6, \\ G_{k+1} &= (G_k ⓩ H)^3. \end{aligned}$$

Fact. G_k is a $(D^{6(k+1)}, D^6, 1/4)$-graph.

Proof.

The base case is clear from Lemma, and the induction step is taken care of by the previous lemma. □

Expander Construction I

The time to access to a neighbor of a node is given by the following inductive equation

$$\begin{aligned} time(G_k) &= 3\cdot(time(G_{k-1}) + 2\cdot time(H)) \\ &= 3^k\cdot time(H^2) + (3^{k-1} + \ldots + 3 + 1)\cdot 2\cdot time(H) \\ &= 2^{O(k)} \\ &= \mathtt{poly}(|G_k|). \end{aligned}$$

The time to compute a neighbor is a polynomial of the graph size. We conclude that the expander family is explicit, but not strongly explicit.

The analysis suggests how to reduce $time(G_k)$.

- Define $time(G_k)$ not in terms of $time(G_{k-1})$ but in terms of $time(G_{k/2})$.

Expander Construction II

Let H be a $(D^{12}, D, 1/16)$-graph constructed from $G_{\mathbb{F}}$. Define

$$
\begin{aligned}
G_1 &= H^2, \\
G_k &= (G_{\lceil k/2 \rceil} \otimes G_{\lfloor k/2 \rfloor})^3 \textcircled{z} H.
\end{aligned}
$$

Fact. G_k is a $(D^{12\cdot(2k-1)}, D^2, 7/8)$-graph.

Proof.

The base case is clear from Lemma, and the induction step is taken care of by the previous lemma. □

Expander Construction II

Fact. G_k is a strongly explicit.

There is a $\mathrm{poly}(k)$-time algorithm that upon receiving a label v of a vertex in G_k and an index j in $[2d]$ finds the j-th neighbor of v. $[|\langle n, v, i\rangle| = \mathrm{polylog}(n).]$

$$\begin{aligned}
time(G_k) &= 3 \cdot time(G_{\lceil k/2\rceil}) + 3 \cdot time(G_{\lfloor k/2\rfloor}) + 2 \cdot time(H) \\
&= 2^{O(\log k)} \cdot time(H^2) + \left(\sum_{i=1}^{\log k} 2^{O(i)} + O(1)\right) \cdot 2 \cdot time(H) \\
&= \mathtt{poly}(k) \\
&= \mathtt{polylog}(|G_k|)。
\end{aligned}$$

The time to compute a neighbor is $\mathrm{poly}(n)$. The expander family is strongly explicit.

Expander Construction II

Suppose $D^{12\cdot(2k+1)} < h < D^{12\cdot(2(k+1)+1)}$ 的 h. We will define an expander with h vertices.

Let $D^{12\cdot(2(k+1)+1)} = xh + r$. Define F_h as follows:

1. Classify $D^{12\cdot(2(k+1)+1)}$ nodes into h groups, with r groups having $x+1$ nodes and $h-r$ groups having x nodes.
2. Think of every group as a single node. Since $D^{12\cdot(2(k+1)+1)}/D^{12\cdot(2k+1)} = D^{24}$, each node has no more than $D^2(x+1) \leq D^{26}$ edges. Add enough self-loops so that the graph is of D^{26}-degree.

Theorem. $\{F_h\}_{h\in\omega}$ is a strongly explicit $\left(D^{26}, \frac{1}{16D^{50}}\right)$-edge expander family.

Reingold's Theorem

Theorem. UPATH $\in \mathbf{L}$.

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1. O. Reingold. Undirected ST-Connectivity in Log-Space. STOC 2005.

The Idea

Connectivity Algorithm for d-degree expander graph is easy.

- The diameter of an expander graph is of length $O(\log(n))$.
- An exhaustive search can be carried out in $O(\log^2(n))$ space.

Reingold's idea draws inspiration from the construction that simulates a random string of length $\log^2(n)$ by a random walk of length $O(\log(n))$ in an expander.

1. Transforming conceptually the input graph G to a graph G_m so that a connected component in G turns to an expander in G_m and unconnected vertices in G remain unconnected in G_m.
2. A neighbor of a vertex in the imaginary G_m can be guessed in constant space.

The Algorithm

Fix the $(D^6, D, 1/4)$-graph H constructed previously. Let (G, s, t) be the input.

1. Convert the input graph G to a D^6-degree graph G_0 on the fly.
   1.1 Replace a large degree vertex by a cycle to decrease degree to no more than 3.
   1.2 Add self-loops to make it degree D^6.

   Let s_0 be a copy of s and t_0 be a copy of t.
2. Repeat the construction $G_k = (G_{k-1} ⓩ H)^3$ on the fly for $m = O(\log |G|)$ times.
   Let s_k be a node in the "cloud" over s_{k-1} and t_k be in the "cloud" over t_{k-1}.
3. Enumerate walks in G_m of length $\ell = O(\log |G|)$, and check if there are some s_m over s_0 and some t_m over t_0 such that s_m connects to t_m.

Correctness.

- s_k and t_k are connected if and only if s_{k-1} and t_{k-1} are connected, inductively.

The Expansion Ratio

Fact. $\{G_k\}_k$ is a (D^6, λ)-expander family for some constant $\lambda \in (0,1)$.

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1. $\lambda_{G_k} < \left(\frac{7}{9}\right)^2$. Then $\lambda_{G_k ⓩ H} \leq 1 - \gamma_H^2 \gamma_{G_k} \leq 1 - \frac{9}{16}\gamma_{G_k} < \frac{7}{9}$. So $\lambda_{G_{k+1}} = (\lambda_{G_k ⓩ H})^3 < \left(\frac{7}{9}\right)^2$.
2. $\lambda_{G_k} \geq \left(\frac{7}{9}\right)^2$. It is easy to derive that $\left(1 - \frac{9}{16}\gamma_{G_k}\right)^2 \leq 1 - \gamma_{G_k} = \lambda_{G_k}$. Consequently

$$\lambda_{G_{k+1}} = \left(1 - \frac{9}{16}\gamma_{G_k}\right)^3 < \lambda_{G_k}^{3/2}.$$

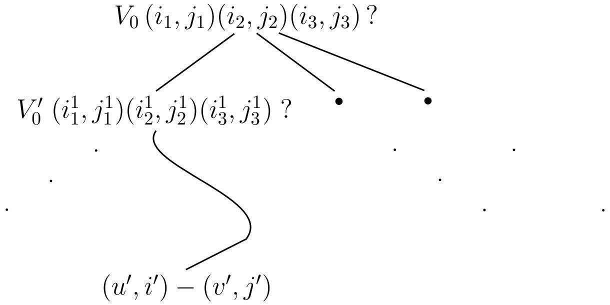
Conclusion: either $\lambda_{G_m} < \left(\frac{7}{9}\right)^2$, or $\lambda_{G_m} < (\lambda_{G_0})^{(3/2)^m} = (\lambda_{G_0})^{\mathtt{poly}(|G|)}$. The latter implies that

$$\lambda_{G_m} < \left(1 - \frac{1}{12|G|^2}\right)^{\mathtt{poly}(|G|)} \leq \lambda$$

for some $\lambda < 1$.

The Data Structure

The algorithm imagines a tree structure for G_m, and exhausts all paths starting from s by carrying out depth first traversal of the imaginary tree repeatedly.



If r is the (i, j)-th neighbor of s and t is the (i', j')-th neighbor of r, then $(i, j)(i', j')$ is stored in the record for $s\ G_m\ t$. The algorithm only stores the current vertex for backtracking.

The Complexity

The space complexity is about accessing a neighbor of a vertex in G_m.

1. G_0 can be constructed in logspace.
2. To visit a neighbor of a node in G_m makes use of the rotation matrix of G_{m-1} three times. The key point is the following.

   $$space\left((G_{m-1}®H)^3\right) \quad = \quad space(G_{m-1}) + O(1).$$

   The size of the additional space depends only on D, which is a constant.
3. The depth first tree traversal keeps a stack of depth bounded by $O(\log n)$.

Lewis and Papadimitriou introduced $\mathbf{SL}$ as the class of problems solvable in logspace by an NTM that satisfies the following.

1. If the answer is 'yes,' one or more computation paths accept.
2. If the answer is 'no,' all paths reject.
3. If the machine can make a transition from configuration C to configuration D, then it can also goes from D to C.

Theorem. `UPATH` is $\mathbf{SL}$-complete.

Corollary. `UPATH` is $\mathbf{L}$-complete.

Proof.

Reingold Theorem implies that $\mathbf{L} = \mathbf{SL}$. □

The problem "$\mathbf{RL} = \mathbf{L}$" is open. The best we know is $\mathbf{RL} \subseteq \mathbf{L}^{3/2}$.