XIII. Turing Reducibility

Yuxi Fu

BASICS, Shanghai Jiao Tong University

The problem with m-reduction is that it imposes too strong a restriction on the use of a result obtained by revoking a subroutine.

Synopsis

1. Relative Computability
2. Turing Reduction
3. Jump Operator
4. Use Principle
5. Modulus Lemma and Limit Lemma

1. Relative Computability

Computation with Oracle

Suppose $\mathcal{O}$ is a total unary function.

Informally a function f is computable relative to $\mathcal{O}$, or $\mathcal{O}$-computable, if f can be computed by an algorithm that is effective in the usual sense, except from time to time during computation f is allowed to consult the oracle function $\mathcal{O}$.

If f is computable in $\mathcal{O}$, the degree of undecidability of f is no more than that of $\mathcal{O}$.

Partial Recursive Function with Oracle

Formally an $\mathcal{O}$-partial recursive function f is constructed from the initial functions and $\mathcal{O}$ by substitution, primitive recursion and minimization.

URM with Oracle

A URM with Oracle, URMO for short, can recognize a fifth kind of instruction, $O(n)$, for every $n \geq 1$.

If $\mathcal{O}$ is the oracle function, then the effect of $O(n)$ is to replace the content r_n of R_n by $\mathcal{O}(r_n)$.

Turing Machine with Oracle

A Turing Machine with Oracle, TMO for short, has an additional read only oracle tape.

If $\mathcal{O}$ is the oracle function, then the oracle tape is preloaded with the string of 0's and 1's that represents $\mathcal{O}$.

In the above definition it is convenient to restrict to those oracles that are characteristic functions.

Numbering URMO Programs

We fix an effective enumeration of all URMO programs

$$P_0^{\mathcal{O}}, P_1^{\mathcal{O}}, P_2^{\mathcal{O}}, \ldots.$$

It is important to notice that the Gödel number of an oracle machine is independent of any specific oracle function.

Notation and Terminology

$P_e^{\mathcal{O}}$ is the e-th URMO.

$\phi_e^{\mathcal{O},n}$ is the n-ary function $\mathcal{O}$-computed by $P_e^{\mathcal{O}}$.

$\phi_e^{\mathcal{O},1}$ is simplified to $\phi_e^{\mathcal{O}}$.

$W_e^{\mathcal{O}}$ is $dom(\phi_e^{\mathcal{O}})$.

$E_e^{\mathcal{O}}$ is $rng(\phi_e^{\mathcal{O}})$.

$\mathcal{C}^{\mathcal{O}}$ is the set of all $\mathcal{O}$-computable functions.

Relative Computability

Fact.

(i) $\mathcal{O} \in \mathcal{C}^{\mathcal{O}}$.

(ii) $\mathcal{C} \subseteq \mathcal{C}^{\mathcal{O}}$.

(iii) If $\mathcal{O}$ is computable, then $\mathcal{C} = \mathcal{C}^{\mathcal{O}}$.

(iv) $\mathcal{C}^{\mathcal{O}}$ is closed under substitution, recursion and minimalisation.

(v) If ψ is a total function that is $\mathcal{O}$-computable, then $\mathcal{C}^{\psi} \subseteq \mathcal{C}^{\mathcal{O}}$.

Relative S-m-n Theorem

Relative S-m-n Theorem. For all $m, n \geq 1$ there is an injective primitive recursive $(m+1)$-ary function $s_n^m(e, \widetilde{x})$ such that for each oracle $\mathcal{O}$ the following holds:

$$\phi_e^{\mathcal{O},m+n}(\widetilde{x}, \widetilde{y}) \simeq \phi_{s_n^m(e,\widetilde{x})}^{\mathcal{O},n}(\widetilde{y}).$$

Notice that $s_n^m(e, \widetilde{x})$ does not refer to $\mathcal{O}$.

Relative Enumeration Theorem

Relative Enumeration Theorem. For each n, the universal function $\psi_U^{\mathcal{O},n}$ for n-ary $\mathcal{O}$-computable functions given by

$$\psi_U^{\mathcal{O},n}(e,\widetilde{x}) \simeq \phi_e^{\mathcal{O},n}(\widetilde{x})$$

is $\mathcal{O}$-computable.

Relative Recursion Theorem

Relative Recursion Theorem. Suppose $f(y,z)$ is a total $\mathcal{O}$-computable function. There is a primitive recursive function $n(z)$ such that for all z

$$\phi^{\mathcal{O},n}_{f(n(z),z)}(\widetilde{x}) \simeq \phi^{\mathcal{O},n}_{n(z)}(\widetilde{x}).$$

Relative Theory

Once we have the three foundational theorems, we can do the recursion theory relative to an oracle function.

A proof of a proposition relativizes if essentially it is also a proof of the relativized proposition.

$\mathcal{O}$-Recursive Set and $\mathcal{O}$-r.e. Set

A is $\mathcal{O}$-recursive if its characteristic function c_A is $\mathcal{O}$-computable.

A is $\mathcal{O}$-r.e. if its partial characteristic function χ_A is $\mathcal{O}$-computable.

$\mathcal{O}$-Recursive Set and $\mathcal{O}$-r.e. Set

Theorem. The following hold.

(i) A is $\mathcal{O}$-recursive iff A and $\overline{A}$ are $\mathcal{O}$-r.e.

(ii) The following are equivalent.

- A is $\mathcal{O}$-r.e.
- $A = W_e^{\mathcal{O}}$ for some e.
- $A = E_e^{\mathcal{O}}$ for some e.
- $A = \emptyset$ or A is the range of a total $\mathcal{O}$-computable function.
- For some $\mathcal{O}$-decidable predicate $R(x, y)$, $x \in A$ iff $\exists y.R(x, y)$.

(iii) $K^{\mathcal{O}} \stackrel{\text{def}}{=} \{x \mid x \in W_x^{\mathcal{O}}\}$ is $\mathcal{O}$-r.e. but not $\mathcal{O}$-recursive.

Computability Relative to Set

Computability relative to a set A means computability relative to its characteristic function c_A.

We write ϕ_e^A for $\phi_e^{c_A}$.
We say A-computability rather than c_A-computability.

We write $f \leq_T A$ to indicate that f is A-computable.

2. Turing Reduction

Turing Reducibility

A set A is Turing reducible to B, or is recursive in B, notation $A \leq_T B$, if $c_A \leq_T B$.

The sets A, B are Turing equivalent, notation $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$.

$A <_T B$ if $A \leq_T B$ and $B \not\leq_T A$.

Turing Completeness

An r.e. set C is (Turing) complete if $A \leq_T C$ for every r.e. set A.

Property of Turing Reducibility

Fact.

(i) $\leq_T$ is reflexive and transitive.

(ii) $\equiv_T$ is an equivalence relation.

(iii) If $A \leq_m B$ then $A \leq_T B$.

(iv) $A \equiv_T \overline{A}$ for all A.

(v) If A is recursive, then $A \leq_T B$ for all B.

(vi) If B is recursive and $A \leq_T B$, then A is recursive.

(vii) If A is r.e. then $A \leq_T K$.

Turing Degree, or Degree of Unsolvability

The equivalence class $d_T(A) = \{B \mid B \equiv_T A\}$ is called the (Turing) degree of A.

Let **D** be the set of all Turing degrees. **D** is an upper semi-lattice.

Turing Degree

The set of Turing degrees is ranged over by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$.

$\mathbf{a} \leq \mathbf{b}$ iff $A \leq_T B$ for some $A \in \mathbf{a}$ and $B \in \mathbf{b}$.

$\mathbf{a} < \mathbf{b}$ iff $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$.

Turing Degree

Theorem. Every pair $\mathbf{a}, \mathbf{b}$ have a unique least upper bound.

Recursive Degree and Recursively Enumerable Degree

A degree containing a recursive set is called a recursive degree.

A degree containing an r.e. set is called an r.e. degree.

Theorem.

(i) There is precisely one recursive degree $\mathbf{0}$, which consists of all the recursive sets and is the unique minimal degree.

(ii) Let $\mathbf{0}'$ be the degree of K.
Then $\mathbf{0} < \mathbf{0}'$ and $\mathbf{0}'$ is the maximum of all r.e. degrees.

Post's Question

In his 1944 paper, Post raised the following question:

$$\exists \mathbf{a}.\ \mathbf{0} < \mathbf{a} < \mathbf{0}'\ ?$$

3. Jump Operator

Relative Recursive Enumerability

A set A is recursively enumerable in B if $\chi_A \leq_T B$.

Lemma. A is r.e. in B iff A is r.e. in $\overline{B}$.

Lemma. $A \leq_T B$ iff both A and $\overline{A}$ are r.e. in B.

Lemma. Suppose B is recursively enumerable in C. If $C \leq_T D$, then B is recursively enumerable in D.

We say that $\mathbf{a}$ is recursively enumerable in $\mathbf{b}$ if some $A \in \mathbf{a}$ is recursively enumerable in some $B \in \mathbf{b}$.

Jump Operator

The jump K^A of A, notation A', is defined by

$$A' = \{x \mid x \in W_x^A\}.$$

The n-th jump:

$$\begin{array}{rcl} A^{(0)} & = & A, \\ A^{(n+1)} & = & (A^{(n)})'. \end{array}$$

Jump Theorem. The following hold:

(i) A' is r.e. in A.

(ii) $A \leq_T A' \not\leq_T A$. (in fact $\overline{A}, A \leq_1 A'$)

Proof.

(i) Given x calculate $\phi_x^A(x)$. If $\phi_x^A(x)\downarrow$ then output 1.

(ii) Using the Relative S-m-n Theorem one constructs an injective primitive recursive function $s(x)$ such that

$$\phi_{s(x)}^A(y) = \begin{cases} y, & \text{if } x \in A (\text{or } x \notin A); \\ \uparrow, & \text{otherwise.} \end{cases} \tag{1}$$

Clearly $x \in A$ iff $s(x) \in A'$. Hence $\overline{A}, A \leq_1 A'$.

Had $A' \leq_T A$, one would be able to construct an A-recursive function that is different from any A-recursive function, which is a contradiction. □

Jump Theorem. The following hold:

(iii) A is r.e. in B iff $A \leq_1 B'$.

(iv) $A \leq_T B$ iff $A' \leq_1 B'$. Consequently $A \equiv_T B$ iff $A' \equiv_1 B'$.

Proof.

(iii) Suppose A is r.e. in B. Using the Relative S-m-n Theorem, one gets an injective recursive function $s(x)$ such that

$$\phi^B_{s(x)}(y) \simeq \textit{if } x \in A \textit{ then } y \textit{ else } \uparrow .$$

Clearly $x \in A$ iff $s(x) \in B'$. Hence $A \leq_1 B'$.

Conversely if $d : A \leq_1 B'$ then $\chi_A(x)$ can be B-computed by "*if* $\phi^B_{d(x)}(d(x)) \downarrow$ *then* 1 *else* $\uparrow$".

(iv) This follows from (i,ii,iii) immediately. □

Beyond R.E. Degree

The jump of $\mathbf{a}$, notation $\mathbf{a}'$, is defined by $d_T(A')$ for some $A \in \mathbf{a}$.

By definition $\mathbf{0}'$ is the jump of $\mathbf{0}$. Hence the infinite hierarchy

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \mathbf{0}''' < \ldots < \mathbf{0}^{(n)} < \ldots.$$

Notice that $\mathbf{0} = d_T(\emptyset)$ and $\mathbf{0}^{(n)} = d_T(\emptyset^{(n)})$.

4. Use Principle

String as Subset

A finite string σ of 0's and 1's can be seen as an initial segment of a characteristic function.

We write $\sigma \subset A$ if $\sigma \subset c_A$ when both are treated as graphs.

Let $|\sigma|$ denote the length of σ.

Let $A{\restriction}x$ be $\{y \in A \mid y < x\}$.

Similarly one defines $\sigma{\restriction}x$.

Use Function

We write $\phi_{e,s}^{A}(x) = y$ if

- $e, x, y < s$;
- $P_s^A(x)$ outputs y in $< s$ steps;
- only numbers $< s$ are tested for membership of A.

The use function $u(A; e, x, s)$ is "1 + the maximum number tested for membership of A during the computation of $\phi_{e,s}^{A}(x)$" if $\phi_{e,s}^{A}(x)\downarrow$; and $u(A; e, x, s)$ is 0 if $\phi_{e,s}^{A}(x)\uparrow$.

The use function $u(A; e, x)$ is $u(A; e, x, s)$ for some s such that $u(A; e, x, s)\downarrow$.

Use Function

$\phi^{\sigma}_{e,s}(x)$ and $\phi^{A\upharpoonright u}_{e,s}(x)$ are defined accordingly.

$\phi^{\sigma}_{e}(x) = y$ if $\exists s.\phi^{\sigma}_{e,s}(x) = y$.

We shall also use notations like $W^{A}_{e,s}$, $W^{\sigma}_{e,s}$, W^{σ}_{e}.

Use Principle

Theorem. The following hold:

(i) $\phi_e^A(x) = y$ implies $\exists s.\exists \sigma \subset A.\phi_{e,s}^{\sigma}(x) = y$.

(ii) $\phi_{e,s}^{\sigma}(x) = y$ implies $\forall t \geq s.\forall \tau \supseteq \sigma.\phi_{e,t}^{\tau}(x) = y$.

(iii) $\phi_e^{\sigma}(x) = y$ implies $\forall A \supseteq \sigma.\phi_e^A(x) = y$.

The Use Principle implies the following

$$(\phi_{e,s}^A(x) = y \wedge A{\restriction}u = B{\restriction}u) \Rightarrow \phi_{e,s}^B(x) = y,$$

where $u = u(A; e, x, s)$.

5. Modulus Lemma and Limit Lemma

Degrees $\leq_T \mathbf{0}'$

We are mainly interested in degrees $\leq_T \mathbf{0}'$, and particularly in the r.e. degrees.

We provide some alternative characterizations of such degrees.

Modulus of Convergence

1. A sequence $\{f_s(x)\}_{s\in\omega}$ of total functions is recursive if there is a recursive function $\widehat{f}(s,x)$ such that $f_s(x) = \widehat{f}(s,x)$ for all s, x.

2. The sequence $\{f_s(x)\}_{s\in\omega}$ converges pointwise to $f(x)$, notation $f = \lim_s f_s$, if for each x, $f_s(x) = f(x)$ for all but finitely many s.

3. A modulus of convergence for the sequence $\{f_s(x)\}_{s\in\omega}$ is a function $m(x)$ such that $f_s(x) = f(x)$ for all $s \geq m(x)$.

Modulus Lemma. Suppose A is r.e. and $f \leq_T A$. Then there are (1) a recursive sequence $\{f_s\}_{s\in\omega}$ such that (2) $f = \lim_s f_s$ and (3) a modulus m of $\{f_s\}_{s\in\omega}$ that is recursive in A.

Proof.

Suppose A is r.e. and $f = \phi_e^A$. Let $A_s = W_{i,s}$ for some $W_i = A$. Define a converging family $\{f_s\}_{s\omega}$ by

$$f_e(x) \;\; = \;\; \begin{cases} \phi_{e,s}^{A_s}(x), & \text{if } \phi_{e,s}^{A_s}(x)\downarrow, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\{f_s\}_{s\in\omega}$ is a recursive sequence. Define m by

$$m(x) = \mu s.\exists z \leq s.(\phi_{e,s}^{A_s\restriction z}(x)\downarrow \wedge A_s\restriction z = A\restriction z).$$

Now m is A-recursive, and by Use Principle is a modulus since $\phi_{e,s}^{A_s\restriction z}(x) = \phi_{e,s}^{A\restriction z}(x) = \phi_e^A(x) = f(x)$ for $s \geq m(x)$. □

Limit Lemma. $f \leq_T A'$ iff there is an A-recursive sequence $\{f_s\}_{s\in\omega}$ such that $f = \lim_s f_s$.

Proof.

Suppose $f \leq_T A'$. Since A' is r.e. in A, the A-recursive sequence $\{f_s\}_{s\in\omega}$ exists by Relative Modulus Lemma.

Suppose $f = \lim_s f_s$ for an A-recursive sequence $\{f_s\}_{s\in\omega}$. Define

$$A_x = \{s \mid \exists t.(s \leq t \wedge f_t(x) \neq f_{t+1}(x))\},$$

which is finite. Now $m(x) = \mu s.(s \notin A_x)$ is Turing equivalent to

$$B = \{\langle s, x\rangle \mid s \in A_x\},$$

which is r.e. in A. Hence $m \equiv_T B \leq_T A'$.

Conclude that $f \leq_T A'$ since $f(x) = f_{m(x)}(x)$. □

Constructing Degrees below $\mathbf{0}'$

Corollary. A function f has degree $\leq \mathbf{0}'$ (meaning $f \leq_T \emptyset'$) iff $f = \lim_s f_s$ for some recursive sequence $\{f_s\}_{s\in\omega}$.

Constructing R.E. Degrees

Corollary. A function f has r.e. degree iff f is the limit of a recursive sequence $\{f_s\}_{s\in\omega}$ that has a modulus $m \leq_T f$.

Proof.

If $f \equiv_T A$ for some r.e. set A, then by Modulus Lemma $m \leq_T A \equiv_T f$.

Suppose $f = \lim_s f_s$ with modulus $m \leq_T f$. As in the proof of the Limit Lemma, $f \leq_T m$. □