XIV. Arithmetic Hierarchy

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We introduce a hierarchy of sets in terms of logical formula and prove its relationship to the hierarchy $\mathbf{0}, \mathbf{0}', \mathbf{0}'', \ldots$ of Turing degree.

Synopsis

1. Arithmetic Hierarchy
2. Post Theorem
3. Σ_n-Complete Set
4. Relative Arithmetic Hierarchy

Arithmetic Hierarchy

Arithmetic Hierarchy

A set B is in Σ_0 (Π_0) if B is recursive.

A set B is in Σ_n, where $n \geq 1$, if there is a recursive relation $R(x, y_1, y_2, \ldots, y_n)$ such that

$$x \in B \text{ iff } \exists y_1.\forall y_2.\exists y_3.\ldots.\mathbf{Q}_n y_n.R(x, y_1, y_2, \ldots, y_n).$$

A set B is in Π_n, where $n \geq 1$, if there is a recursive relation $R(x, y_1, y_2, \ldots, y_n)$ such that

$$x \in B \text{ iff } \forall y_1.\exists y_2.\forall y_3.\ldots.\mathbf{Q}_n y_n.R(x, y_1, y_2, \ldots, y_n).$$

$\Delta_n = \Sigma_n \cap \Pi_n$.

Arithmetic Set

B is arithmetical if $B \in \bigcup_{n \in \omega} (\Sigma_n \cup \Pi_n)$.

Basic Property

Theorem. The following hold:

(i) $A \in \Sigma_n$ iff $\overline{A} \in \Pi_n$.

(ii) If $A \in \Sigma_n$ (Π_n) then $\forall m{>}n.A \in \Sigma_m \cap \Pi_m$.

(iii) If $A, B \in \Sigma_n$ (Π_n) then $A \cup B, A \cap B \in \Sigma_n$ (Π_n).

(iv) If $R \in \Sigma_n \wedge n > 0 \wedge A = \{x : (\exists y)R(x,y)\}$ then $A \in \Sigma_n$.

(v) If $B \leq_m A \wedge A \in \Sigma_n$ then $B \in \Sigma_n$.

(vi) If $R \in \Sigma_n$ (Π_n) and A, B are defined by

$$\begin{aligned} \langle x, y \rangle \in A &\Leftrightarrow \forall z < y.R(x,y,z), \\ \langle x, y \rangle \in B &\Leftrightarrow \exists z < y.R(x,y,z), \end{aligned}$$

then $A, B \in \Sigma_n$ (Π_n).

$Fin \in \Sigma_2$

Fact. $Fin \in \Sigma_2$.

$$
\begin{aligned}
x \in Fin &\Leftrightarrow W_x \text{ is finite} \\
&\Leftrightarrow \exists s.\forall t.\,(t \leq s \vee W_{x,t} = W_{x,s})\,.
\end{aligned}
$$

Fact. $Inf \in \Pi_2$.

$Cof \in \Sigma_3$

Fact. $Cof \in \Sigma_3$.

$$\begin{aligned} x \in Cof &\Leftrightarrow \overline{W_x} \text{ is finite} \\ &\Leftrightarrow \exists y.\forall z.(z \leq y \vee z \in W_x) \\ &\Leftrightarrow \exists y.\forall z.\exists s.(z \leq y \vee z \in W_{x,s}). \end{aligned}$$

$Tot \in \Pi_2$

Fact. $\{\langle x, y\rangle \mid W_x \subseteq W_y\} \in \Pi_2$.

$$
\begin{aligned}
W_x \subseteq W_y &\Leftrightarrow \forall z. (z \in W_x \Rightarrow z \in W_y) \\
&\Leftrightarrow \forall z. (z \notin W_x \vee z \in W_y) \\
&\Leftrightarrow \forall z. (\forall s. z \notin W_{x,s} \vee \exists t. z \in W_{y,t}) \\
&\Leftrightarrow \forall z. \forall s. \exists t. (z \notin W_{x,s} \vee z \in W_{y,t}) \\
&\Leftrightarrow \forall w. \exists t. \left((w)_0 \notin W_{x,(w)_1} \vee (w)_0 \in W_{y,t}\right).
\end{aligned}
$$

Fact. $\{\langle x, y\rangle \mid W_x = W_y\} \in \Pi_2$.

Fact. $Tot = \{x \mid W_x = \omega\} \in \Pi_2$.

$Rec \in \Sigma_3$

Fact. $Rec \in \Sigma_3$.

$$
\begin{aligned}
x \in Rec &\Leftrightarrow W_x \text{ is recursive} \\
&\Leftrightarrow \exists y.\left(W_x = \overline{W_y}\right) \\
&\Leftrightarrow \exists y.\,(W_x \cap W_y = \emptyset \wedge W_x \cup W_y = \omega) \\
&\Leftrightarrow \exists y.\,((\forall s.W_{x,s} \cap W_{y,s} = \emptyset) \wedge (\forall z.\exists s.z \in W_{x,s} \cup W_{y,s}))\,.
\end{aligned}
$$

$Ext \in \Sigma_3$

Fact. $Ext \in \Sigma_3$.

$$
\begin{aligned}
x \in Ext \quad &\Leftrightarrow \quad \exists y.\,(\phi_x \subseteq \phi_y \wedge W_y = \omega) \\
&\Leftrightarrow \quad \exists y.\forall z.\exists s.\exists t.\,((z \notin W_{x,s} \vee \phi_{x,s}(z) = \phi_{y,s}(z)) \wedge z \in W_{y,t})\,.
\end{aligned}
$$

$Crt \in \Sigma_3$

Fact. $Crt = \{x \mid W_x \text{ is creative}\} \in \Sigma_3$.

$$
\begin{aligned}
x \in Crt &\Leftrightarrow \overline{W_x} \text{ is productive} \\
&\Leftrightarrow \exists y.\forall z.\left(W_z \subseteq \overline{W_x} \Rightarrow (\phi_y(z)\downarrow \wedge \phi_y(z) \in \overline{W_x} \setminus W_z)\right) \\
&\Leftrightarrow \exists y.\forall z.(W_z \cap W_x = \emptyset \Rightarrow (\phi_y(z)\downarrow \wedge \phi_y(z) \notin W_x \cup W_z)) \\
&\Leftrightarrow \exists y.\forall z.(W_z \cap W_x \neq \emptyset \vee (\phi_y(z)\downarrow \wedge \phi_y(z) \notin W_x \cup W_z))
\end{aligned}
$$

Now $W_z \cap W_x \neq \emptyset$ iff

$$\exists s.W_{z,s} \cap W_{x,s} \neq \emptyset,$$

and $\phi_y(z)\downarrow \wedge \phi_y(z) \notin W_x \cup W_z$ iff

$$\exists s.z \in W_{y,s} \wedge \forall s.(z \notin W_{y,s} \vee \phi_{y,s}(z) \notin W_{x,s} \cup W_{z,s}).$$

Let P_{TM} be $\{x \mid P_x \text{ runs in polynomial time}\}$.

$$x \in P_{TM} \quad \Leftrightarrow \quad \exists c.\forall z.(P_x(z) \text{ terminates in } cz^c)$$

Hence $P_{TM} \in \Sigma_2$.

Post Theorem

Completeness

A set $A \in \Sigma_n$ is Σ_n-complete if $B \leq_1 A$ for every $B \in \Sigma_n$.

A set $A \in \Pi_n$ is Π_n-complete if $B \leq_1 A$ for every $B \in \Pi_n$.

Post Theorem

(i) $B \in \Sigma_{n+1}$ iff B is r.e. in a Π_n set iff B is r.e. in a Σ_n set.

Proof.

If $B \in \Sigma_{n+1}$, then $x \in B$ iff $\exists y.R(x,y)$ for some $R \in \Pi_n$. So B is r.e. in $\{\langle x,y\rangle \mid R\} \in \Pi_n$.

Suppose B is r.e. in some $C \in \Pi_n$. Then for some e,

$$x \in B \text{ iff } x \in W_e^C \text{ iff } \exists s.\exists \sigma.(\sigma \subset C \wedge x \in W_{e,s}^{\sigma}).$$

Now $x \in W_{e,s}^{\sigma}$ is recursive, and $\sigma \subset C$ is C-recursive since

$$\sigma \subset C \quad \text{iff} \quad \forall y < |\sigma|.(\sigma(y) = 1 \wedge y \in C \vee \sigma(y) = 0 \wedge y \notin C).$$

Hence $B \in \Sigma_{n+1}$. □

Post Theorem

(ii) $\emptyset^{(n)}$ is Σ_n-complete for all $n > 0$.

Proof.

$\emptyset' = K$ is Σ_1-complete.

Now assume $\emptyset^{(n)}$ is Σ_n-complete. Then

$$\begin{aligned} B \in \Sigma_{n+1} \quad & \text{iff} \quad B \text{ is r.e. in some } \Sigma_n \text{ set} \\ & \text{iff} \quad B \text{ is r.e. in } \emptyset^{(n)} \\ & \text{iff} \quad B \leq_1 \emptyset^{(n+1)}. \end{aligned}$$

Hence $\emptyset^{(n+1)}$ is Σ_{n+1}-complete. □

Post Theorem

(iii) $B \in \Sigma_{n+1}$ iff B is r.e. in $\emptyset^{(n)}$.

(iv) $B \in \Delta_{n+1}$ iff $B \leq_T \emptyset^{(n)}$.

Proof.

We have the following equivalence:

$$\begin{aligned} B \in \Delta_{n+1} \quad & \text{iff} \quad B, \overline{B} \in \Sigma_{n+1} \\ & \text{iff} \quad B, \overline{B} \text{ are r.e. in } \emptyset^{(n)} \\ & \text{iff} \quad B \leq_T \emptyset^{(n)}. \end{aligned}$$

□

Hierarchy Theorem. $\forall n > 0. \Delta_n \subset \Sigma_n \wedge \Delta_n \subset \Pi_n$.

Proof.
$\emptyset^{(n)} \in \Sigma_n \setminus \Pi_n$ and $\overline{\emptyset^{(n)}} \in \Pi_n \setminus \Sigma_n$. □

A Comment on Completeness

$$
\begin{aligned}
B \leq_m \emptyset^{(n)} &\Rightarrow B \in \Sigma_n \\
&\Rightarrow B \text{ is r.e. in } \emptyset^{(n-1)} \\
&\Rightarrow B \leq_1 \emptyset^{(n)} \\
&\Rightarrow B \leq_m \emptyset^{(n)}.
\end{aligned}
$$

The following is the relativized version of "$K \leq_m A$ iff $K \leq_1 A$":

$$\emptyset^{(n)} \leq_m A \text{ iff } \emptyset^{(n)} \leq_1 A.$$

Σ_n-Complete Set

Let (A_1, A_2) and (B_1, B_2) be two pairs of sets such that $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$.

Then $(A_1, A_2) \leq_m (B_1, B_2)$ if there is a recursive function f such that $f(A_1) \subseteq B_1$, $f(A_2) \subseteq B_2$ and $f(\overline{A_1 \cup A_2}) \subseteq \overline{B_1 \cup B_2}$.

We write $(A_1, A_2) \leq_1 (B_1, B_2)$ if f is one-one.

For $n > 0$ we write $(\Sigma_n, \Pi_n) \leq_m (C, D)$ if $(A, \overline{A}) \leq_m (C, D)$ for some Σ_n-complete set A. The notation $(\Sigma_n, \Pi_n) \leq_1 (C, D)$ is defined similarly.

Fin is Σ_2-Complete, *Tot* is Π_2-Complete

Theorem. $(\Sigma_2, \Pi_2) \leq_1 (Fin, Tot)$.

Proof.

$Fin \in \Sigma_2$ and $Tot \in \Pi_2$.

Let A be in Σ_2. There is a recursive relation R such that

$$x \in \overline{A} \text{ iff } \forall y.\exists z.R(x, y, z).$$

By S-m-n Theorem there is a one-one recursive function s s.t.

$$\phi_{s(x)}(u) = \begin{cases} 0, & \text{if } \forall y \leq u.\exists z.R(x, y, z), \\ \uparrow, & \text{otherwise.} \end{cases}$$

Now $x \in \overline{A} \Rightarrow W_{s(x)} = \omega \Rightarrow s(x) \in Tot$

and $x \in A \Rightarrow W_{s(x)}$ is finite $\Rightarrow s(x) \in Fin$. □

Cof and *Rec* are Σ_3-Complete

Let *Cmp* be $\{x \mid W_x \equiv_T K\}$, the set of Turing complete r.e. sets.

Theorem. $(\Sigma_3, \Pi_3) \leq_1 (\mathit{Cof}, \mathit{Cmp}) \leq_1 (\mathit{Rec}, \mathit{Cmp})$.

Corollary. *Cof* is Σ_3-complete.

Corollary. (Rogers) *Rec* is Σ_3-complete.

$(\Sigma_3, \Pi_3) \leq_1 (\mathit{Cof}, \mathit{Cmp})$

Fix an $A \in \Sigma_3$. Then some $R \in \Pi_2$ exists such that

$$x \in A \text{ iff } \exists y.R(x, y).$$

Since *Inf* is Π_2-complete, a one-one recursive function g exists s.t.

$$R(x, y) \text{ iff } W_{g(x,y)} \text{ is infinite.}$$

We will construct an r.e. set $W_{f(x)} = \bigcup_{s \in \omega} W^s_{f(x)}$ in stages s.t.

$$x \in A \text{ iff } W_{f(x)} \text{ is cofinite.}$$

$(\Sigma_3, \Pi_3) \leq_1 (Cof, Cmp)$

Let the elements of the cofinite set $\overline{W^s_{f(x)}}$ be denoted by

$$b^s_{x,0} < b^s_{x,1} < b^s_{x,2} < \ldots < b^s_{x,k} < \ldots.$$

Let $W^0_{f(x)} := \emptyset$.

Let $W^{s+1}_{f(x)} := W^s_{f(x)}$. Additionally put $b^s_{x,k}$ in $W^{s+1}_{f(x)}$ if $k \leq s$ and

$$W_{g(x,k),s} \neq W_{g(x,k),s+1} \vee k \in K_{s+1} \setminus K_s.$$

So we have constructed some programme $P_{f(x)}$ that enumerates $W_{f(x)}$, from which we can calculate $f(x)$.

$(\Sigma_3, \Pi_3) \leq_1 (Cof, Cmp)$

If $x \in A$, then $W_{g(x,k)}$ is infinite for some k; and $|\overline{W_{f(x)}}| \leq k$.

If $x \notin A$, then $W_{g(x,k)}$ is finite for all k. There is a stage when the first $k+1$ elements $b_{x,0} < b_{x,1} < b_{x,2} < \ldots < b_{x,k}$ of $\overline{W_{f(x)}}$ have all been fixed. So $\overline{W_{f(x)}}$ is infinite. Conclude that $A \leq_1 Cof$.

To prove $\overline{A} \leq_1 Cmp$, we show that if $x \notin A$ then $K \leq_T W_{f(x)}$.

For each k we can $W_{f(x)}$-recursively calculate a stage $s(k)$ such that $b_{x,k}^{s(k)} = b_{x,k}$. Notice that $k \in K$ iff $k \in K_{s(k)}$.

Relative Arithmetic Hierarchy

Relative Post Theorem

Relative Post Theorem. For every $n \geq 0$, the following hold:

(i) $A^{(n+1)}$ is Σ^A_{n+1}-complete.

(ii) $B \in \Sigma^A_{n+1}$ iff B is r.e. in $A^{(n)}$.

(iii) $B \leq_T A^{(n)}$ iff $B \in \Delta^A_{n+1}$.

Low Degree and High Degree

A degree $\mathbf{a} \leq \mathbf{0}'$ is low if $\mathbf{a}' = \mathbf{0}'$.

A degree $\mathbf{a} \leq \mathbf{0}'$ is high if $\mathbf{a}' = \mathbf{0}''$.

Low Degree

Theorem. For $A \leq_T \emptyset'$, the following are equivalent:

(i) A is low.

(ii) $\Sigma_1^A \subseteq \Pi_2$.

(iii) $A' \leq_1 \overline{\emptyset^{(2)}}$.

Proof.

The following equivalences hold:

$$\begin{array}{lll} A \text{ is low} & \text{iff} & A' \leq_T \emptyset' \\ & \text{iff} & A' \in \Delta_2, \text{ Post Theorem,} \\ & \text{iff} & \Sigma_1^A \subseteq \Delta_2, \ A' \text{ is } \Sigma_1^A \text{ complete,} \\ & \text{iff} & \Sigma_1^A \subseteq \Pi_2, \ \Sigma_1^A \subseteq \Sigma_1^{\emptyset'} = \Sigma_2, \\ & \text{iff} & A' \leq_1 \overline{\emptyset^{(2)}}, \ \overline{\emptyset^{(2)}} \text{ is } \Pi_2 \text{ complete.} \end{array}$$

□

High Degree

Theorem. For $A \leq_T \emptyset'$, the following are equivalent:

(i) A is high.

(ii) $\Sigma_2 \subseteq \Pi_2^A$.

(iii) $\emptyset^{(2)} \leq_1 \overline{A^{(2)}}$.

Proof.

The following equivalences hold:

$$\begin{aligned} A \text{ is high} \quad & \text{iff} \quad \emptyset'' \leq_T A' \\ & \text{iff} \quad \emptyset'' \in \Delta_2^A, \\ & \text{iff} \quad \Sigma_2 \subseteq \Delta_2^A,\ \emptyset'' \text{ is } \Sigma_2 \text{ complete}, \\ & \text{iff} \quad \Sigma_2 \subseteq \Pi_2^A,\ \Sigma_2 \subseteq \Sigma_2^A, \\ & \text{iff} \quad \emptyset^{(2)} \leq_1 \overline{A^{(2)}},\ \overline{A^{(2)}} \text{ is } \Pi_2^A \text{ complete}. \end{aligned}$$

□