III. Church's Lambda Calculus

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Origin

Foundation of mathematics was very much an issue in the early decades of 20th century.

Cantor, Frege, Russel's Paradox, Principia Mathematica, NBG/ZF

Combinatory Logic and λ-Calculus were originally proposed as part of foundational systems, which are based on a concept of function rather than a concept of set.

Combinatory Logic

Schönfinkel [4Sep.1889-1942, Russian] proposed Combinatory Logic as a variable free presentation of functions [1924].

von Neumann [28Dec.1903-8Feb.1957] used a combinatory notation in his formulation of set theory.

Curry [12Sep.1900-1Sep.1982] reinvented Combinatory Logic in an effort to formalize the notion of substitution.

Lambda Calculus

Alonzo Church [14Jun.1903-11Aug.1995] invented the λ-calculus with a foundational motivation [1932].

The ambition to provide a foundation for mathematics failed after the discovery of Kleene-Rosser Paradox.

As a foundation for computation and programming, the λ-calculus has been extremely successful.

What is CL/λ About

CL/λ was proposed to describe the basic properties of function abstraction, application, substitution.

In λ the concept of abstraction is taken as primitive.

In CL it is defined in terms of more primitive operators.

Synopsis

1. Syntax and Semantics
2. Church-Rosser Property
3. Definability

1. Syntax and Semantics

Lambda Calculus is a functional model of computation.

Syntax

Grammar for λ-term:

$$M := x \mid \lambda x.M \mid M \cdot M',$$

where x is a variable, $\lambda x.M$ is an abstraction term, and $M \cdot M'$ is an application term.

$\lambda x_1.\lambda x_2 \ldots \lambda x_k.M$ is often abbreviated to $\lambda x_1 x_2 \ldots x_k.M$ or $\lambda\widetilde{x}.M$,

$M \cdot M'$ to MM', and

$(\ldots((MM_1)M_2)\ldots M_{k-1})M_k$ to $MM_1M_2\ldots M_K$.

Let $\equiv$ be the syntactic (grammar) equality.

Operational Semantics

Structural Semantics:

$$\begin{array}{rcl} (\lambda x.M)N & \rightarrow & M\{N/x\}, \ \ \beta \text{ reduction} \\ MN & \rightarrow & M'N, \text{ if } M \rightarrow M', \ \ \text{structural rule} \\ MN & \rightarrow & MN', \text{ if } N \rightarrow N', \ \ \text{eager evaluation} \\ \lambda x.M & \rightarrow & \lambda x.M', \text{ if } M \rightarrow M', \ \ \text{partial evaluation.} \end{array}$$

Let $\rightarrow^*$ be the reflexive and transitive closure of $\rightarrow$.

The β-conversion relation $=$ is the equivalence closure of $\rightarrow^*$.

Bound Variable, Closed Term

The variable x in $\lambda x.M$ is bound. (α-conversion)
A variable in a term is free if it is not bound. (notation $fv(M)$)

A λ-term is closed if it contains no free variables.

Redex

The following reductions make use of α-conversion:

$$\begin{array}{rcl}(\lambda xy.yxx)((\lambda uv.v)y) & \rightarrow & \lambda z.z((\lambda uv.v)y)((\lambda uv.v)y) \\ & \rightarrow & \lambda z.z(\lambda v.v)((\lambda uv.v)y) \\ & \rightarrow & \lambda z.z(\lambda v.v)(\lambda v.v).\end{array}$$

An alternative evaluation strategy:

$$\begin{array}{rcl}(\lambda xy.yxx)((\lambda uv.v)y) & \rightarrow & (\lambda xy.yxx)(\lambda v.v) \\ & \rightarrow & \lambda y.y(\lambda v.v)(\lambda v.v).\end{array}$$

A subterm of the form $(\lambda x.M)N$ is called a redex, and $M\{N/x\}$ a reduct of the reduction that contracts the redex.

λ-Term as Proof

A combinator is a closed λ-term.
Some famous combinators are:

$$\begin{aligned}\mathbf{I} &\stackrel{\text{def}}{=} \lambda x.x,\\ \mathbf{K} &\stackrel{\text{def}}{=} \lambda xy.x,\\ \mathbf{S} &\stackrel{\text{def}}{=} \lambda xyz.xz(yz).\end{aligned}$$

It is easy to see that $\mathbf{I} = \mathbf{SKK}$.
Logical interpretation of $\mathbf{I}, \mathbf{K}, \mathbf{S}$.

Theorem. $\forall M \in \Lambda^0.\exists L \in \{\mathbf{K}, \mathbf{S}\}^+.L \to^* M$.

A Term that Generates All

Let $\mathbf{X}$ be $\lambda z.z\mathbf{KSK}$. Then $\mathbf{K} = \mathbf{XXX}$ and $\mathbf{S} = \mathbf{X}(\mathbf{XX})$.

Corollary. $\forall M \in \Lambda^0.\exists L \in \{\mathbf{X}\}^+.L = M$.

Normal Form

Let $\mathbf{\Omega}$ be $(\lambda x.xx)(\lambda x.xx)$. Then

$$\mathbf{\Omega} \rightarrow \mathbf{\Omega} \rightarrow \mathbf{\Omega} \rightarrow \ldots$$

This is the simplest divergent λ-term.

A λ-term M is in normal form if $M \rightarrow M'$ for no M'.

We will show that the normal form of a term is unique.

Fixpoint

Lemma. $\forall F.\exists X.FX = X$.

Proof. Define the fixpoint combinator $\mathbf{Y}$ by

$$\mathbf{Y} \stackrel{\text{def}}{=} \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)).$$

It is easily seen that $F(\mathbf{Y}F) = \mathbf{Y}F$.

However it is not the case that $\mathbf{Y}F \rightarrow^* F(\mathbf{Y}F)$.

Fixpoint

The Turing fixpoint $\mathbf{\Theta}$ is defined by AA, where

$$A \stackrel{\text{def}}{=} \lambda xy.y(xxy).$$

Clearly

$$\mathbf{\Theta}F \rightarrow^* F(\mathbf{\Theta}F).$$

Fixpoint

Suppose we need to find some F such that $Fxy = FyxF$.

The equality follows from $F = \lambda xy.FyxF$.

So we may let F be a fixpoint of $\lambda f.\lambda xy.fyxf$.

Reduction Strategy

lazy reduction, head reduction, leftmost reduction, standard reduction, Gross-Knuth reduction, . . .

A reduction $M \to N$ is a head reduction, notation $M \to_h N$, if it is obtained by applying the β-reduction, the structural rule and the partial evaluation rule.

The reflexive and transitive closure of $\to_h$ is denoted by $\to_h^*$.

2. Church-Rosser Property

Church-Rosser Theorem

Although the reduction $\rightarrow$ is nondeterministic, it has the following confluence (diamond, Church-Rosser) property:

- If $M \rightarrow^* M'$ and $M \rightarrow^* M''$, then some M''' exists such that $M' \rightarrow^* M'''$ and $M'' \rightarrow^* M'''$.

In other words the result of evaluating a λ-term is unique.

Proof of Church-Rosser Theorem

Define the reduction $\twoheadrightarrow$ inductively as follows:
(i) $M \twoheadrightarrow M$;
(ii) if $M \twoheadrightarrow M'$ then $\lambda x.M \twoheadrightarrow \lambda x.M'$;
(iii) if $M \twoheadrightarrow M'$ and $N \twoheadrightarrow N'$ then $MN \twoheadrightarrow M'N'$;
(iv) if $M \twoheadrightarrow M'$ and $N \twoheadrightarrow N'$ then $(\lambda x.M)N \twoheadrightarrow M'\{N'/x\}$.

Fact. If $M \twoheadrightarrow M'$ and $N \twoheadrightarrow N'$ then $M\{N/x\} \twoheadrightarrow M'\{N'/x\}$.

Fact. $\twoheadrightarrow$ satisfies the confluence property.

Fact. $\rightarrow^*$ is the transitive closure of $\twoheadrightarrow$.

Implication of Church-Rosser Theorem

Fact. If $M = N$ then $M \rightarrow^* Z$ and $N \rightarrow^* Z$ for some Z.

Fact. If N is a nf of M then $M \rightarrow^* N$.

Fact. Every λ-term has at most one nf.

Fact. If M, N are distinct nf's, then $M \neq N$.

Theorem. The theory λ is consistent.

3. Definability

Church Numeral

Church introduced the following encoding of numbers:

$$c_n \stackrel{\text{def}}{=} \lambda fx.f^n(x).$$

Rosser defined the following arithmetic operations:

$$\mathbf{A}_+ \stackrel{\text{def}}{=} \lambda xypq.xp(ypq),$$
$$\mathbf{A}_\times \stackrel{\text{def}}{=} \lambda xyz.x(yz),$$
$$\mathbf{A}_{exp} \stackrel{\text{def}}{=} \lambda xy.yx.$$

Boolean Term

The Boolean values are encoded by:

$$\textbf{true} \stackrel{\text{def}}{=} \lambda xy.x,$$
$$\textbf{false} \stackrel{\text{def}}{=} \lambda xy.y.$$

The term "*if B then M else N*" is represented by

$$BMN.$$

Pairing

The pairing and projections can be defined as follows:

$$
\begin{aligned}
[M, N] &\stackrel{\text{def}}{=} \lambda z.\textit{if } z \textit{ then } M \textit{ else } N, \\
\pi_0 &\stackrel{\text{def}}{=} \lambda z.z\ \mathbf{true}, \\
\pi_1 &\stackrel{\text{def}}{=} \lambda z.z\ \mathbf{false}.
\end{aligned}
$$

Barendregt Numeral

Barendregt introduced the following encoding of natural numbers:

$$\ulcorner 0 \urcorner \stackrel{\text{def}}{=} \mathbf{I},$$
$$\ulcorner n+1 \urcorner \stackrel{\text{def}}{=} [\mathbf{false}, \ulcorner n \urcorner].$$

We call the normal forms $\ulcorner 0 \urcorner, \ulcorner 1 \urcorner, \ulcorner 2 \urcorner, \ldots$ numerals.

The successor, predecessor and test-for-zero can be defined by

$$\mathbf{S}^{+} \stackrel{\text{def}}{=} \lambda z.[\mathbf{false}, z],$$
$$\mathbf{P}^{-} \stackrel{\text{def}}{=} \lambda z.z\ \mathbf{false},$$
$$\mathbf{Zero} \stackrel{\text{def}}{=} \lambda z.z\ \mathbf{true}.$$

Lambda Definability

A k-ary function f is λ-definable if there is a combinator F such that for all numbers $n_1, \ldots, n_k$ one has the following terminating head reduction path

$$F\ulcorner n_1 \urcorner \ldots \ulcorner n_k \urcorner \quad \rightarrow_h^* \quad \ulcorner f(n_1, \ldots, n_k) \urcorner$$

if $f(n_1, \ldots, n_k)$ is defined, and the following divergent head reduction path

$$F\ulcorner n_1 \urcorner \ldots \ulcorner n_k \urcorner \rightarrow_h \rightarrow_h \rightarrow_h \rightarrow_h \ldots$$

if $f(n_1, \ldots, n_k)$ is undefined.

Numerals are Solvable

Fact. $\forall n. \lceil n \rceil \mathbf{KII} \rightarrow_h^* \mathbf{I}$.

Definability of Initial Function

The zero function, successor function and projection functions are λ-defined respectively by

$$
\begin{aligned}
\mathbf{Z} &\stackrel{\text{def}}{=} \lambda x_1 \ldots x_k.\lceil 0 \rceil, \\
\mathbf{S}^+ &\stackrel{\text{def}}{=} \lambda x.[\mathbf{false}, x], \\
\mathbf{U}_i^k &\stackrel{\text{def}}{=} \lambda x_1 \ldots x_k.x_i.
\end{aligned}
$$

These terms admit only head reduction.

Definability of Composition

Suppose $f, g_1(\widetilde{x}), \ldots, g_k(\widetilde{x})$ are λ-defined by $F, G_1, \ldots, G_k$.

Then $f(g_1(\widetilde{x}), \ldots, g_k(\widetilde{x}))$ is λ-defined by

$$\lambda\widetilde{x}.(G_1\widetilde{x}\mathbf{KII})\ldots(G_k\widetilde{x}\mathbf{KII})F(G_1\widetilde{x})\ldots(G_k\widetilde{x}).$$

Definability of Recursion

Consider the function f defined by the recursion:

$$\begin{array}{rcl} f(\widetilde{x}, 0) & = & h(\widetilde{x}), \\ f(\widetilde{x}, y+1) & = & g(\widetilde{x}, y, f(\widetilde{x}, y)). \end{array}$$

Suppose h, g are λ-defined by H, G respectively.

Intuitively f is λ-defined by F such that

$$F \rightarrow_h^* \lambda \widetilde{x} y. \mathit{if}\, \mathbf{Zero}(y) \textit{ then } H\widetilde{x} \textit{ else } G\widetilde{x}(\mathbf{P}^- y)(F\widetilde{x}(\mathbf{P}^- y)).$$

By the Fixpoint Theorem we may define F by

$$\mathbf{\Theta}\left(\lambda f. \lambda \widetilde{x} y. \mathit{if}\, \mathbf{Zero}(y) \textit{ then } H\widetilde{x} \textit{ else } G\widetilde{x}(\mathbf{P}^- y)(f\widetilde{x}(\mathbf{P}^- y))\right).$$

Definability of Minimization

Let μ_P be defined as follows:

$$\mu_P \stackrel{\text{def}}{=} \mathbf{\Theta}(\lambda hz.\mathit{if}\ Pz\ \mathit{then}\ z\ \mathit{else}\ h[\mathbf{false}, z]).$$

If $P\lceil n \rceil \rightarrow_h^* \mathbf{false}$ for all n, then

$$\begin{array}{rcl} \mu_P\lceil 0 \rceil & \rightarrow_h^* & \mathit{if}\ P\lceil 0 \rceil\ \mathit{then}\ \lceil 0 \rceil\ \mathit{else}\ \mu_P\lceil 1 \rceil \\ & \rightarrow_h^* & \mu_P\lceil 1 \rceil \\ & \rightarrow_h^* & \mathit{if}\ P\lceil 1 \rceil\ \mathit{then}\ \lceil 1 \rceil\ \mathit{else}\ \mu_P\lceil 2 \rceil \\ & \rightarrow_h^* & \mu_P\lceil 2 \rceil \\ & \rightarrow_h^* & \ldots, \end{array}$$

and consequently $\mu_P\lceil 0 \rceil$ is unsolvable.

Definability of Minimization

Suppose $g(\widetilde{x}, z)$ is a total recursive function and $f(\widetilde{x})$ is define by

$$\mu z.(g(\widetilde{x}, z) = 0).$$

Assume that g is λ-defined by G.

Then f is λ-defined by

$$F \stackrel{\text{def}}{=} \lambda\widetilde{x}.\mu_{\lambda z.\mathbf{Zero}(G\widetilde{x}z)}\ulcorner 0 \urcorner.$$

Kleene Theorem

Theorem [Kleene, 1936]. All recursive functions are λ-definable.

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