IX. Recursively Enumerable Set

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We have seen that many sets, although not recursive, can be effectively generated in the sense that, for any such set, there is an effective procedure that produces the elements of the set in a non-stop manner.

We shall formalize this intuition in this lecture.

Synopsis

1. Recursively Enumerable Set
2. Characterization of r.e. Set
3. Rice-Shapiro Theorem
4. Recursive Enumeration of r.e. Set

1. Recursively Enumerable Set

The Definition of Recursively Enumerable Set

The partial characteristic function of a set A is given by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ \uparrow, & \text{if } x \notin A. \end{cases}$$

A is recursively enumerable if χ_A is computable.

We shall often abbreviate 'recursively enumerable set' to 'r.e. set'.

Partially Decidable Problem

A problem $f : \omega \rightarrow \{0, 1\}$ is partially decidable if $dom(f)$ is r.e.

Partially Decidable Predicate

A predicate $M(\widetilde{x})$ of natural number is partially decidable if its partial characteristic function

$$\chi_M(\widetilde{x}) = \begin{cases} 1, & \text{if } M(\widetilde{x}) \text{ holds,} \\ \uparrow, & \text{if } M(\widetilde{x}) \text{ does not hold,} \end{cases}$$

is computable.

$$\begin{aligned}\text{Partially Decidable Problem} &\Leftrightarrow \text{Partially Decidable Predicate} \\ &\Leftrightarrow \text{Recursively Enumerable Set}\end{aligned}$$

Example

1. The halting problem is partially decidable. Its partial characteristic function is given by

$$\chi_H(x,y) = \begin{cases} 1, & \text{if } P_x(y)\downarrow, \\ \uparrow, & \text{otherwise.} \end{cases}$$

2. K, K_0, K_1 are r.e.. But none of $\overline{K}, \overline{K_0}, \overline{K_1}$ is r.e..

Index for Recursively Enumerable Set

A set is r.e. iff it is the domain of a unary computable function.

- So $W_0, W_1, W_2, \ldots$ is an enumeration of all r.e. sets.
- Every r.e. set has an infinite number of indexes.

Closure Property

Union Theorem. The recursively enumerable sets are closed under union and intersection uniformly and effectively.

Proof.

According to S-m-n Theorem there are primitive recursive functions $r(x, y), s(x, y)$ such that

$$\begin{aligned} W_{u(x,y)} &= W_x \cup W_y, \\ W_{i(x,y)} &= W_x \cap W_y. \end{aligned}$$

□

The Most Hard r.e. Set

Fact. If $A \leq_m B$ and B is r.e. then A is r.e..

Theorem. A is r.e. iff $A \leq_1 K$.

Proof.

Suppose A is r.e. Let $f(x, y)$ be defined by

$$f(x, y) = \begin{cases} 1, & \text{if } x \in A, \\ \uparrow, & \text{if } x \notin A. \end{cases}$$

By S-m-n Theorem there is an injective primitive recursive function $s(x)$ st. $f(x, y) = \phi_{s(x)}(y)$. It is clear that $x \in A$ iff $s(x) \in K$. $\square$

Comment. No r.e. set is more difficult than K.

2. Characterization of r.e. Set

Normal Form Theorem

Normal Form Theorem. $M(\widetilde{x})$ is partially decidable iff there is a primitive recursive predicate $R(\widetilde{x}, y)$ such that $M(\widetilde{x})$ iff $\exists y.R(\widetilde{x}, y)$.

Proof.

If $R(\widetilde{x}, y)$ is primitive recursive and $M(\widetilde{x}) \Leftrightarrow \exists y.R(\widetilde{x}, y)$, then the computable function "*if* $\mu y R(\widetilde{x}, y)$ *then* 1 *else* $\uparrow$" is the partial characteristic function of $M(\widetilde{x})$.

Conversely suppose $M(\widetilde{x})$ is partially decided by P. Let $R(\widetilde{x}, y)$ be

$$P(x) \downarrow \text{ in } y \text{ steps.}$$

Then $R(\widetilde{x}, y)$ is primitive recursive and $M(\widetilde{x}) \Leftrightarrow \exists y.R(\widetilde{x}, y)$. □

Quantifier Contraction Theorem

Quantifier Contraction Theorem. If $M(\widetilde{x}, y)$ is partially decidable, so is $\exists y.M(\widetilde{x}, y)$.

Proof.

Let $R(\widetilde{x}, y, z)$ be a primitive recursive predicate such that

$$M(\widetilde{x}, y) \Leftrightarrow \exists z.R(\widetilde{x}, y, z).$$

Then $\exists y.M(\widetilde{x}, y) \Leftrightarrow \exists y.\exists z.R(\widetilde{x}, y, z) \Leftrightarrow \exists u.R(\widetilde{x}, (u)_0, (u)_1)$. □

Uniformisation Theorem

Uniformisation Theorem. If $R(x, y)$ is partially decidable, then there is a computable function $c(x)$ such that $c(x)\downarrow$ iff $\exists y.R(x, y)$ and $c(x)\downarrow$ implies $R(x, c(x))$.

We may think of $c(x)$ as a choice function for $R(x, y)$. The theorem states that the choice function is computable.

A is r.e. iff there is a partially decidable predicate $R(x, y)$ such that $x \in A$ iff $\exists y.R(x, y)$.

Complementation Theorem

Complementation Theorem. A is recursive iff A and $\overline{A}$ are r.e.

Proof.

Suppose A and $\overline{A}$ are r.e. Then some primitive recursive predicates $R(x, y), S(x, y)$ exist such that

$$\begin{aligned} x \in A &\Leftrightarrow \exists y R(x, y), \\ x \in \overline{A} &\Leftrightarrow \exists y S(x, y). \end{aligned}$$

Now let $f(x)$ be $\mu y(R(x, y) \vee S(x, y))$.

Then $f(x)$ is total and computable, and

$$x \in A \Leftrightarrow R(x, f(x)).$$

□

Applying Complementation Theorem

Fact. $\overline{K}$ is not r.e.

Comment. If $\overline{K} \leq_m A$ then A is not r.e. either.

Applying Complementation Theorem

Fact. If A is r.e. but not recursive, then $\overline{A} \not\leq_m A \not\leq_m \overline{A}$.

Comment. However A and $\overline{A}$ are intuitively equally difficult.

Graph Theorem

Graph Theorem. Let $f(x)$ be a partial function. Then $f(x)$ is computable iff the predicate '$f(x) \simeq y$' is partially decidable iff $\{\langle x, y\rangle \mid f(x) \simeq y\}$ is r.e.

Proof.

If $f(x)$ is computable by $P(x)$, then

$$f(x) \simeq y \Leftrightarrow \exists t.(P(x) \downarrow y \text{ in } t \text{ steps}).$$

The predicate '$P(x) \downarrow y$ in t steps' is primitive recursive.

Conversely let $R(x, y, t)$ be such that

$$f(x) \simeq y \Leftrightarrow \exists t.R(x, y, t).$$

Now $f(x) \simeq \mu y.R(x, y, \mu t.R(x, y, t))$. □

Listing Theorem

Listing Theorem. A is r.e. iff either $A = \emptyset$ or A is the range of a unary total computable function.

Proof.

Suppose A is nonempty and its partial characteristic function is computed by P. Let a be a member of A. The total function $g(x,t)$ given by

$$g(x,t) = \begin{cases} x, & \text{if } P(x)\downarrow \text{ in } t \text{ steps}, \\ a, & \text{if otherwise.} \end{cases}$$

is computable. Clearly A is the range of $h(z) = g((z)_1, (z)_2)$.

The converse follows from Graph Theorem. □

Listing Theorem

The theorem gives rise to the terminology ‘recursively enumerable’.

Implication of Listing Theorem

A set is r.e. iff it is the range of a computable function.

Implication of Listing Theorem

Corollary. For each infinite nonrecursive r.e. A, there is an injective total recursive function f such that $rng(f) = A$.

Corollary. Every infinite r.e. set has an infinite recursive subset.

Proof.

Suppose $A = rng(f)$. An infinite recursive subset is enumerated by the total increasing computable function g given by

$$\begin{aligned} g(0) &= f(0), \\ g(n+1) &= f(\mu y(f(y) > g(n))). \end{aligned}$$

□

Applying Listing Theorem

Fact. The set $\{x \mid \phi_x \text{ is total}\}$ is not r.e.

Proof.

If $\{x \mid \phi_x \text{ is total}\}$ were a r.e. set, then it is the range of a total computable function f.

The function $g(x)$ given by $g(x) = \phi_{f(x)}(x) + 1$ would be total and computable. □

3. Rice-Shapiro Theorem

Rice-Shapiro Theorem. Suppose that $\mathcal{A}$ is a set of unary computable functions such that the set $\{x \mid \phi_x \in \mathcal{A}\}$ is r.e. Then for any unary computable function f, $f \in \mathcal{A}$ iff there is a finite function $\theta \subseteq f$ with $\theta \in \mathcal{A}$.

Comment. Intuitively a set of recursive functions is r.e. iff it is effectively generated by an r.e. set of finite functions.

Proof of Rice-Shapiro Theorem

Suppose $A = \{x \mid \phi_x \in \mathcal{A}\}$ is r.e.

($\Rightarrow$): Suppose $f \in \mathcal{A}$ but $\forall$ finite $\theta \subseteq f.\theta \notin \mathcal{A}$.

Let P be a partial characteristic function of K. Define the computable function $g(z,t)$ by

$$g(z,t) \simeq \begin{cases} f(t), & \text{if } P(z) \not\downarrow \text{ in } t \text{ steps,} \\ \uparrow, & \text{otherwise.} \end{cases}$$

According to S-m-n Theorem, there is an injective primitive recursive function $s(z)$ such that $g(z,t) \simeq \phi_{s(z)}(t)$.

By construction $\phi_{s(z)} \subseteq f$ for all z.

$z \in K \Rightarrow \phi_{s(z)}$ is finite $\Rightarrow s(z) \notin A$;
$z \notin K \Rightarrow \phi_{s(z)} = f \Rightarrow s(z) \in A$.

Proof of Rice-Shapiro Theorem

($\Leftarrow$): Suppose f is a computable function and there is a finite $\theta \in \mathcal{A}$ such that $\theta \subseteq f$ and $f \notin \mathcal{A}$.

Define the computable function $g(z,t)$ by

$$g(z,t) \simeq \begin{cases} f(t), & \text{if } t \in Dom(\theta) \vee z \in K, \\ \uparrow, & \text{otherwise.} \end{cases}$$

According to S-m-n Theorem, there is an injective primitive recursive function $s(z)$ such that $g(z,t) \simeq \phi_{s(z)}(t)$.

$z \in K \Rightarrow \phi_{s(z)} = f \Rightarrow s(z) \notin A$;
$z \notin K \Rightarrow \phi_{s(z)} = \theta \Rightarrow s(z) \in A$.

What Rice-Shapiro Theorem cannot Do

Can we apply Rice-Shapiro Theorem to show that any of the following sets is non-r.e.:

$$
\begin{aligned}
Fin &= \{x \mid W_x \text{ is finite}\}, \\
Inf &= \{x \mid W_x \text{ is infinite}\}, \\
Tot &= \{x \mid \phi_x \text{ is total}\}, \\
Con &= \{x \mid \phi_x \text{ is total and constant}\}, \\
Cof &= \{x \mid W_x \text{ is cofinite}\}, \\
Rec &= \{x \mid W_x \text{ is recursive}\}, \\
Ext &= \{x \mid \phi_x \text{ is extensible to a total recursive function}\}.
\end{aligned}
$$

Reversing Rice-Shapiro Theorem

$\{x \mid \phi_x \in \mathcal{A}\}$ is r.e. if the following hold:

1. $\Theta = \{e(\theta) \mid \theta \in \mathcal{A} \text{ and } \theta \text{ is finite}\}$ is r.e., where e is a canonical effective encoding of the finite functions.
2. $\forall f \in \mathcal{A}.\exists \text{ finite } \theta \in \mathcal{A}.\theta \subseteq f$.

Comment. We cannot take e as the Gödel encoding function of the recursive functions. Why? How would you define e?

4. Recursive Enumeration of r.e. Set

We have seen that *Fin* is not r.e., which implies that the Gödel numbers for programs do not make a very useful indexing system for the finite sets.

It is clear however that there is a simple and natural encoding for the finite sets, which can be exploited to define effective approximations of r.e. sets.

Canonical Index for Finite Set

Suppose $A = \{x_1, \ldots, x_k\}$, where $x_1 < \ldots < x_k$.

The number $2^{x_1} + \ldots + 2^{x_k}$ is the canonical index of A.

Let 0 be the canonical index of the empty set.

Let D_y denote the finite set with canonical index y.

There are recursive functions m and s such that

$$\begin{aligned} m(x) &= \max(D_x), \\ s(x) &= |D_x|. \end{aligned}$$

Strong Array

A sequence $\{A_n\}_{n\in\omega}$ of finite sets is a strong array if some recursive function f exists such that $A_n = D_{f(n)}$ for all $n \in \omega$.

A strong array $\{A_n\}_{n\in\omega}$ is disjoint if $A_m \cap A_n = \emptyset$ whenever $m \neq n$. It is cumulative if $A_n \subseteq A_{n+1}$ for all $n \in \omega$.

Approximation of r.e. Set

Lemma. For each infinite r.e. set A, there is an infinite number of disjoint/cumulative strong array $\{A_n\}_{n\in\omega}$ such that

$$A = \bigcup_{n\in\omega} A_n.$$

The lemma, and its proof, suggest the next definition.

Recursive Enumeration

A recursive enumeration, or simply an enumeration, of an r.e. set A consists of a strong array $\{A_s\}_{s\in\omega}$ such that $A_s \subseteq A_{s+1}$ for all $s \in \omega$ and

$$A = \bigcup_{s\in\omega} A_s.$$

A Standard Enumeration of R.E. Set

Recall that $e, x, y, t < s$ whenever $s = \langle e, x, y, t \rangle$.

Let $\phi_{e,s}(x)$ be defined by

$$
\begin{array}{rcl}
\phi_{e,0}(x) & = & \bot, \\
\phi_{e,s+1}(x) & = & \left\{ \begin{array}{ll} y, & \text{either } \phi_{e,s}(x) = y, \text{or } P_e(x) \text{ outputs } y \\ & \text{in } t \text{ steps for } t > 0 \text{ st. } s = \langle e, x, y, t \rangle, \\ \bot, & \text{otherwise.} \end{array} \right.
\end{array}
$$

Let $W_{e,s}$ be the domain of $\phi_{e,s}$.

$W_{e,0} \subseteq W_{e,1} \subseteq \ldots \subseteq W_{e,s} \subseteq \ldots$ is a recursive enumeration.

Property of $\phi_{e,s}$ and $W_{e,s}$

1. $(\phi_{e,s}(x) = y) \Rightarrow (e, x, y < s)$.
2. $\forall s.\exists$ at most one $\langle e, x, y\rangle.(\phi_{e,s}(x) = y) \wedge (\phi_{e,s-1}(x) \uparrow)$.
3. $\forall s.\exists$ at most one $\langle e, x\rangle.x \in W_{e,s+1} \setminus W_{e,s}$.
4. $\{\langle e, x, s\rangle \mid \phi_{e,s}(x) \neq \bot\}$ is recursive.
5. $\{\langle e, x, y, s\rangle \mid \phi_{e,s}(x) = y\}$ is recursive.